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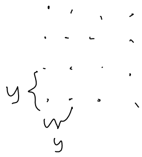
$$\bigcup_{n=1}^{\infty} [n^3, n^3+n]$$

Recall: SZ: Any $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ contains $\{x, x+y, x+2y, \dots, x+ky\}$

<u>Thm (Furstenberg-Katznelson)</u>: Let $A \subset \mathbb{N}^d$ with $\bar{d}(A) = \limsup_{N \to \infty} \frac{|A \cap \{1, \dots, N\}^d|}{N^d} > 0$

Then for any finite set $F \subset \mathbb{N}^d$ $\exists$ $x \in \mathbb{N}^d$ and $y \in \mathbb{N}$ s.t. $\underbrace{x + yF}_{\{x + yv \,:\, v \in F\}} \subset A$

<u>Example</u>: $F = \{1, \dots, k\}^{\cancel{d}\,2} \subset \mathbb{N}^2$



Exercise: Use Sz thm to get rectangular grids.

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Thm (F.K.): Let $(X, \mathcal{B}, \mu)$ be a prob. space and let $T_1, T_2, \dots, T_k$ be commuting m.p. transformations on $(X, \mathcal{B}, \mu)$. Then $\forall$ $A \in \mathcal{B}$ with $\mu(A) > 0$, $\exists n \in \mathbb{N}$ s.t.

$$\mu\left(A \cap T_1^{-n} A \cap T_2^{-n} A \cap \dots \cap T_k^{-n} A\right) > 0$$

Taking $T_i = T^i$ we recover Furstenberg's multiple recurrence thm.

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Thm (Bergelson-Leibman): Let $p_1, p_2, \dots, p_k \in \mathbb{Z}[x]$ and $p_i(0)=0$. Then for any $A \subset \mathbb{N}$ with $\bar{d}(A)>0$, $\exists\, n, x \in \mathbb{N}$ s.t.

$$\{x, x+p_1(n), x+p_2(n), \dots, x+p_k(n)\} \subset A.$$

Thm: Let p_i be as above. Then if $\mathbb{N} = C_1 \cup \dots \cup C_r$, $\exists\, i \in \{1, \dots, r\}$ and $x, n \in \mathbb{N}$ s.t.

$$\{x, x+p_1(n), \dots, x+p_k(n)\} \subset C_i$$

Furstenberg - Katznelson - Ornstein

Furstenberg, 1981, book

Einsiedler - Ward, book, "Ergodic theory with a view towards Number theory"

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Topological Dynamics

Def: A topological (dynamical) system is a pair (X,T) where X is a compact metric space and $T: X \to X$ homeomorphism.

Def: A system (X,T) is minimal if there is no $Y \subset X$ closed, non-empty and invariant under T (i.e. $TY \subset Y$).

Obs: Any system (X,T) contains a minimal subsystem (Y,T).

$Y \subset X$, closed

Prop: If (X,T) is minimal, then any $A \subset X$ open and $A \neq \phi$ recurs, i.e., $\exists n \in \mathbb{N}$ s.t. $A \cap T^{-n} A \neq \phi$

! $X = \bigcup_{n=1}^{\infty} T^{-n} A$

Pf: Let $Y = \left[\bigcup_{n=1}^{\infty} T^{-n} A\right]^C$ is closed

$Y \neq X$, but Y is T-invariant.

$\Rightarrow Y = \phi$ so $\exists n$ s.t. $T^{-n} A \cap A \neq \phi$.

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Recall: Given $E \subset \mathbb{N}$ $\exists$ m.p.s. $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ s.t. $\mu(A) = \bar{d}(E)$ and $\forall n_1, \dots, n_k$

$$\bar{d}\left(E \cap (E - n_1) \cap \dots \cap (E - n_k)\right) \geq \mu(A \cap T^{-n_1} A \cap \dots \cap T^{-n_k} A).$$

Thm: Suppose $\mathbb{N} = C_1 \cup \dots \cup C_r$. Then $\exists$ minimal system (X, T) and an open cover $X = D_1 \cup \dots \cup D_r$ s.t. for any $n_1, \dots, n_k \in \mathbb{N}$, and $i \in \{1, \dots, r\}$, $D_i \cap T^{-n_1} D_i \cap \dots \cap T^{-n_k} D_i \neq \emptyset \Rightarrow C_i \cap (C_i - n_1) \cap \dots \cap (C_i - n_k) \neq \emptyset$

Pf: Let $\chi: \mathbb{N} \to \{1, \dots, r\}$ be $\chi = \sum i \mathbb{1}_{C_i}$. Let $X_0 = \{1, \dots, r\}^{\mathbb{N}_0}$ is a compact metric space, let $T: X_0 \to X_0$ be the left shift. Let $B_i = \{x \in X_0 : x(0) = i\}$ for each $i = 1, \dots, r$.

Let $X_1 = \overline{\{T^n \chi : n \in \mathbb{N}\}}$. Note X_1 is T-invariant. Let $X \subset X_1$ be a minimal subsystem.

Let $D_i = B_i \cap X$. If $y \in D_i \cap T^{-n_1} D_i \cap \dots \cap T^{-n_k} D_i$, then $\exists m \in \mathbb{N}$ s.t. $y(n_j) = T^m \chi(n_j)$

$\Rightarrow m \in C_i \cap (C_i - n_1) \cap \dots \cap (C_i - n_k)$.

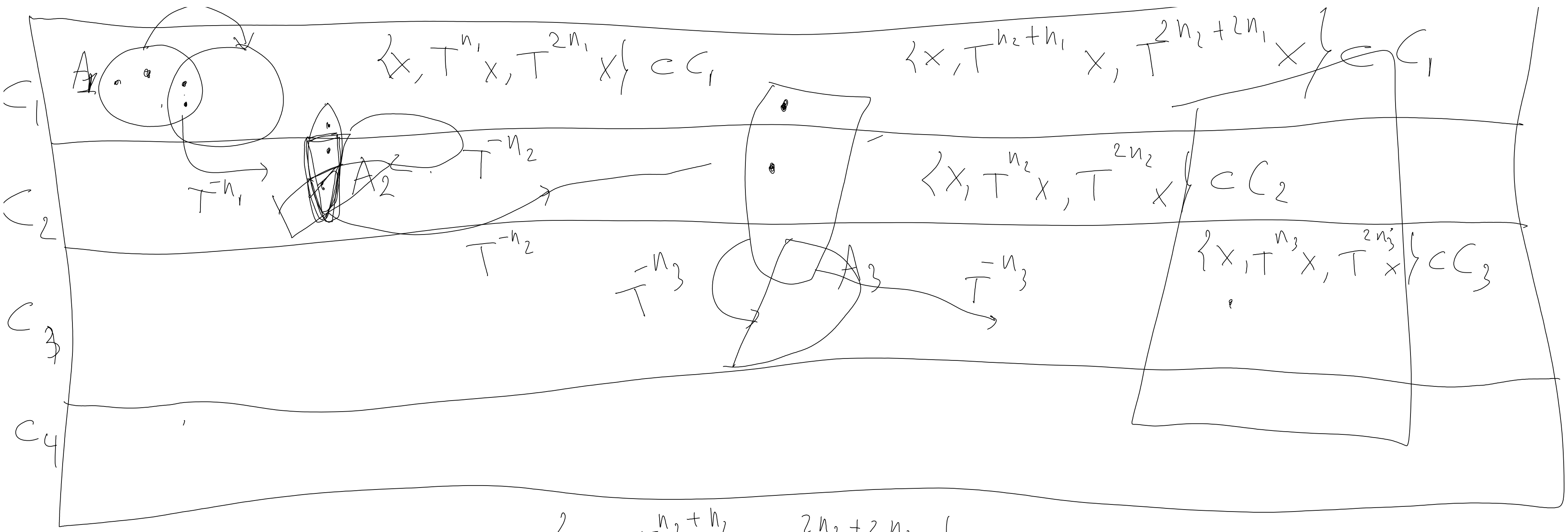
Recall (vdW thm): If $\mathbb{N} = C_1 \cup \dots \cup C_r$, $\forall k \in \mathbb{N}$, $\exists i \in \{1, \dots, r\}$ and $x, n \in \mathbb{N}$ s.t. $\{x, x+n, x+2n, \dots, x+kn\} \subset C_i$

Thm: Let (X, T) be a minimal system, suppose $X = C_1 \cup \dots \cup C_r$, C_i is open, then $\forall k \in \mathbb{N}$ $\exists i \in \{1, \dots, r\}$ and $n \in \mathbb{N}$ s.t. $C_i \cap T^{-n} C_i \cap T^{-2n} C_i \cap \dots \cap T^{-kn} C_i \neq \emptyset$.

Pf for $k = 2$:

T^{-n_1}

$$A_2 = C_2 \cap T^{-n_1}\left(A_1 \cap T^{-n_1} A_1\right)$$



$\{x, T^{n_3+n_2}x, T^{2n_3+2n_2}x\} \subset C_2$

Proposition: Let ~~$n_1, n_2, \dots, n_k$~~ $k \in \mathbb{N}$. TFAE

(1) Whenever (X,T) is minimal and $X = C_1 \cup \dots \cup C_r$ one C_i satisfies $C_i \cap T^{-n_1} C_i \cap \dots \cap T^{-n_k} C_i \neq \emptyset$

(2) Whenever (X,T) is minimal and $A \subset X$ is open, $A \neq \emptyset$, $\exists\, i \in \mathbb{N}$ s.t. $T^{-i}A \cap T^{-n_1-i}A \cap \dots \cap T^{-n_k-i}A \neq \emptyset$.

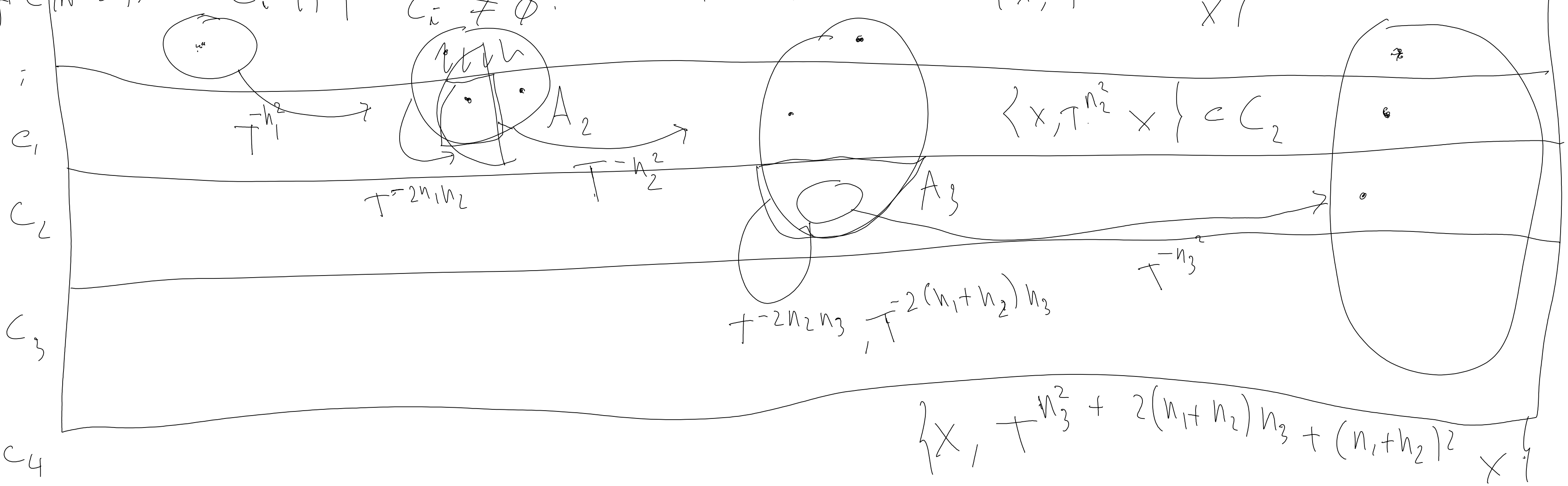
Proposition: Let $k \in \mathbb{N}$. TFAE

(*) If (X,T) minimal and $X = C_1 \cup \cdots \cup C_r$, some C_i satisfies $\underbrace{C_i \cap T^{-n} C_i \cap \cdots \cap T^{-kn} C_i}$ $\neq \emptyset$ for some n

$\Leftarrow$ If (X,T) minimal and $A \subset X$ open, $A \neq \emptyset$, $\exists\, n \in \mathbb{N}$ s.t. $A \cap T^{-n} A \cap \cdots \cap T^{-kn} A \neq \emptyset$.

Thm: If $\mathbb{N} = C_1 \cup \cdots \cup C_r$, some $C_i \supset \{x, x+y^2\}$

Thm: If (X,T) is minimal and $X = C_1 \cup \cdots \cup C_r$ is an open cover, then, $\exists\, i \in \{1, \ldots, r\}$ and $n \in \mathbb{N}$ s.t. $C_i \cap T^{-n^2} C_i \neq \emptyset$.



Recall: A set $S \subset \mathbb{N}$ is called syndetic if $\exists\, M$ s.t. $S \cup (S-1) \cup \cdots \cup (S-M) = \mathbb{N}$.

Def: A set $T \subset \mathbb{N}$ is thick if $\forall M \in \mathbb{N} \; \exists N \in \mathbb{N}$ s.t. $\{N, N+1, \dots, N+M\} \subset T$.

~~Exercise~~: T is thick iff $\mathbb{N} \setminus T$ is not syndetic.

Ex: T is thick iff $\forall S \subset \mathbb{N}$ syndetic, $T \cap S \neq \phi$.

S is syndetic iff $\forall T \subset \mathbb{N}$ thick, $T \cap S \neq \phi$

Def: $A \subset \mathbb{N}$ is piecewise syndetic if $A = S \cap T$ for some syndetic S and thick T.