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Recall: $S \subset \mathbb{N}$ is <u>syndetic</u> if $\exists r \in \mathbb{N}$ s.t. $S \cup (S-1) \cup \cdots \cup (S-r) = \mathbb{N}$

$T \subset \mathbb{N}$ is thick if $\forall N \in \mathbb{N}\ \exists m \in \mathbb{N}$ s.t. $\{m, m+1, \dots, m+N\} \subset T$

Ex: If S is syndetic and T is thick, then $S \cap T \neq \phi$

S is syndetic iff $\forall T \subset \mathbb{N}$ thick, $S \cap T \neq \phi$.

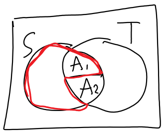
T is thick iff $\forall S$ syndetic $S \cap T \neq \phi$.

S is syndetic iff $\mathbb{N} \setminus S$ is not thick; T is thick iff $\mathbb{N} \setminus T$ is not syndetic.

Def: $A \subset \mathbb{N}$ is piecewise syndetic if $A = S \cap T$ for some syndetic S and thick T.

Lemma (Brown's lemma): If A is PWS and $A = A_1 \cup \cdots \cup A_r$ then at least one of the A_i is PWS.

Pf:



Let $\tilde{S} = S \setminus A_2$. If $\tilde{S}$ is syndetic, $A_1 = \tilde{S} \cap T$ is PWS

If $\tilde{T} := \mathbb{N} \setminus \tilde{S}$ is thick, $A_2 = \tilde{T} \cap S$ is PWS

Recall: A top. dyn. system is (X,T) where X is compact metric space and $T: X \to X$ is continuous. A system is minimal if there are no proper closed and T-invariant subsets of X.

Ex: (X,T) is minimal iff every $x \in X$ has a dense orbit. $\{x, Tx, T^2x, \dots\}$

Def: (X,T) is transitive if $\exists x \in X$ s.t. $\overline{\{x, Tx, T^2x, \dots\}} = X$.

Given a system (X,T), a set $U \subset X$ and $x \in X$, denote $V(x,U) = \{n \in \mathbb{N} : T^n x \in U\}$

Lemma: A system (X,T) is minimal iff $\forall$ open $U \neq \phi$ and $\forall x \in X$, $V(x,U)$ is syndetic.

Pf: If $V(x,U)$ is non-empty $\forall U, \forall x$, then every orbit is dense, hence (X,T) is minimal.

Conversely, if (X,T) is minimal and $U \subset X$ is open, $U \neq \phi$, then $Y = X \setminus \bigcup_{n=0}^{\infty} T^{-n} U$ is closed T-inv. and not X. Hence $Y = \phi$, so by compactness, $\exists r \in \mathbb{N}$ s.t. $X = \bigcup_{n=0}^{r} T^{-n} U$.

Let $S = V(x,U)$ for some $x \in X$. Let $m \in \mathbb{N}$. Then $T^m x \in T^{-n} U$ for some $n \in \{0, \dots, r\}$, therefore $T^{n+m} x \in U$, so $n+m \in S$. Therefore S is syndetic

Ex: If (X,T) is transitive, then (X,T) minimal $\Leftrightarrow \forall U \subset X$ open, $\forall x \in U$, $V(x,U)$ is syndetic.

Def: Say $R \subset \mathbb{N}$ is a set of top. recurrence if $\forall$ minimal (X,T) and $\forall U \subset X, U \neq \phi, \exists n \in R$

s.t. $U \cap T^{-n} U \neq \emptyset$.

Thm: Let $R \subset \mathbb{N}$. TFAE

(1) R is a set of top. recurrence.

(2) If $\mathbb{N} = C_1 \cup \cdots \cup C_r$, $\exists\, C \in \{C_1, \dots, C_r\}$ s.t. $(C-C) \cap R \neq \emptyset$

(3) For every top. dyn. system (X,T), if $X = C_1 \cup \cdots \cup C_r$ for open C_i, $\exists\, C \in \{C_1, \dots, C_r\}$ and $n \in R$ s.t. $C \cap T^{-n} C \neq \emptyset$

(4) For every syndetic $S \subset \mathbb{N}$, $(S-S) \cap R \neq \emptyset$.

(5) For every PWS $A \subset \mathbb{N}$, $(A-A) \cap R \neq \emptyset$.

(6) For every PWS $A \subset \mathbb{N}$ $\exists\, n \in R$ s.t. $A \cap (A-n)$ is PWS.

~~Thm (Kriz)~~: $\exists$ R of top. rec. that is not a set of recurrence.

Def: Call a pattern to a collection P of finite subsets of $\mathbb{N}$. An element $C \in P$ is called a ~~configuration~~.

A pattern P is shift invariant if $\forall\, C \in P$ $\forall\, n \in \mathbb{N}$, $C + n \in P$

Ex: $P = \{\{x, x+y, x+2y\} : x,y \in \mathbb{N}\}$ is a shift-invariant pattern. | $P = \{\{x, x+y^2\} : x,y \in \mathbb{N}\}$

$P = \{\{x, y, x+y\} : x,y \in \mathbb{N}\}$ is not shift-invariant.

A pattern P is ~~monochromatic~~ if whenever $\mathbb{N} = C_1 \cup \cdots \cup C_r$, $\exists\, C \in P$ and $i \in \{1, \dots, r\}$ s.t. $C \subseteq C_i$

Thm: Let P be a shift-invariant pattern. TFAE.

(2) P is monochromatic

(1) $\forall$ minimal (X,T), $\forall\, U \subset X$ open, $U \neq \emptyset$, $\exists\, C \in P$ s.t. $\bigcap_{n \in C} T^{-n} U \neq \emptyset$.

(3) $\forall$ system (X,T), if $X = C_1 \cup \cdots \cup C_r$, C_i open, $\exists\, C \in P$, $\exists\, i \in \{1, \dots, r\}$ s.t. $\bigcap_{n \in C} T^{-n} C_i \neq \emptyset$.

(4) $\forall$ syndetic $S \subset \mathbb{N}$, $\exists\, C \in P$ s.t. $C \subseteq S$.

(5) $\forall$ PWS $A \subset \mathbb{N}$, $\exists\, C \in P$ s.t. $C \subseteq A$

(6) $\forall$ PWS $A \subset \mathbb{N}$, $\exists\, c \in P$ s.t. $\underbrace{\{n : n + C \subseteq A\}}_{\bigcap_{i \in C} A - i}$ is PWS.

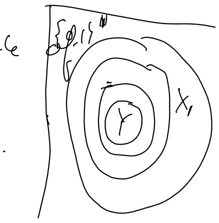
Remark: Let $R \subset \mathbb{N}$. Letting $P = \{\{x, x+n\} : x \in \mathbb{N}, n \in R\}$ we recover the previous thm.

If $C_i \cap (C_i - n_1) \cap \cdots \cap (C_i - n_k) \neq \emptyset$ and $\{n_1, \dots, n_k\} \in P$, then $C_i \supset A$ for some $A \in P$

$(1) \Leftarrow (2) \Leftrightarrow (3)$ $\qquad (2) \Rightarrow (4)$. $\qquad (5) \Rightarrow (2)$ $\qquad (6) \Rightarrow (5)$ $\qquad (4) \Rightarrow (1)$

Lemma: Let (X,T) be a top. dyn. system. Let $x \in X$ with dense orbit. Let $U \subset X$ be open and suppose $\exists$ minimal $Y \subset X$ s.t. $U \cap Y \neq \emptyset$. Then $V(x,U)$ is PWS.

~~Let P be a monot pattern,~~ Let $A \subset \mathbb{N}$ be PWS. Let $w = 1_A \in \{0,1\}^{\mathbb{N}}$, let T be left-shift.
Let $X_1 = \overline{\{T^n 1_A : n \in \mathbb{N}\}}$. Let $A = S \cap T$ for S syndetic and T thick. Let $m_N \in \mathbb{N}$ s.t. $\{m_N, m_N+1, \dots, m_N+N\} \subset T \ \forall N$. Let y be the limit of a subsequence $\{T^{m_N} 1_A\}$. $y \in X_1$. Letting $\tilde{S} = \{n : y(n) = 1\}$; $\tilde{S}$ is syndetic.



Exercise! Let Y be minimal subsystem of $\overline{\{T^n y : n \in \mathbb{N}\}}$

Claim: $(0,0,0,\dots) \notin \overline{\{T^n y : n \in \mathbb{N}\}}$. (Exercise!)

Let $U = \{x \in X_1 : x_0 = 1\}$. Apply (1) to find $C \in P$ s.t. $W := \bigcap_{n \in C} T^{-n} U \neq \emptyset$.
$W \cap Y \neq \emptyset$ so we can apply lemma and get $B = \{n : T^n 1_A \in W\}$ is PWS.

If $m \in B$ then $T^m 1_A \in W$ so $1_A \in U \ \forall n \in C \Leftrightarrow 1_A(n+m) = 1 \ \forall n \in C$
$\Leftrightarrow m + C \subseteq A$. $\blacksquare$

Thm: If $\mathbb{N} = C_1 \cup \cdots \cup C_r$, some $C \in \{C_1, \dots, C_r\}$ contains $\{x+y, xy\}$.

Obs: (1) The pattern $\{\{x+y, xy\} : x, y \in \mathbb{N}\}$ is not shift-invariant.

(2) Not every set $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$ contains $\{x+y, xy\}$.

(3) $\{x+y, xy\}$ involves both addition ~~and multiplication~~ D_n.
$x \mapsto nx$

Let $T_n : \mathbb{N} \to \mathbb{N}$
$x \mapsto x + n$ $\qquad D_m \circ T_n = T_{nm} \circ D_m$ $\qquad$ (Distributivity)

Obs: $A \supset \{x+y, xy\} \Leftrightarrow y \in \boxed{T_x^{-1} A \cap D_x^{-1} A}$

Thm (vdW): Let $A \subset \mathbb{N}$ be PWS. Then ~~$\exists n$~~ $\forall k \in \mathbb{N} \ \exists n \in \mathbb{N}$ s.t.

$$A \cap (A-n) \cap (A-2n) \cap \cdots \cap (A-kn) \text{ is PWS}$$

Ex: If A is PWS then $D_n A$ is PWS.

Pf: WLOG, C_1 is PWS.

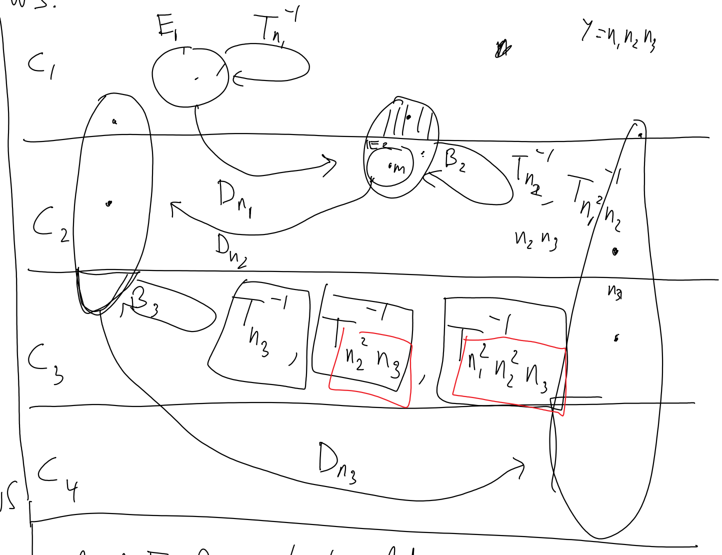
$\exists\, n_1 \in \mathbb{N}$ s.t. $E_1 = C_1 \cap C_1 - n_1$ is PWS.

If $n_1 E_1 \cap C_1 \neq \emptyset$, let $m \in E_1$ s.t. $\underline{m n_1 \in C_1}$. Then $\underline{m + n_1 \in C_1}$.

Otherwise WLOG $B_2 := n_1 E_1 \cap C_2$ is PWS. Using vdW, find $n_2 \in \mathbb{N}$ s.t.

$E_2 := B_2 \cap (B_2 - n_2) \cap (B_2 - \boxed{n_1^2 n_2})$ is PWS

If $n_2 E_2 \cap C_2 \neq \emptyset$, let $m \in E_2$ s.t. $n_2 m \in C_2$. Then $n_2 + m \in C_2$



If $n_2 E_2 \cap C_1 \neq \emptyset$, let $m \in E_2$ s.t. $\underline{n_2 m \in C_1}$. Then $m + n_1^2 n_2 \in B_2$, hence

$x + n_1 n_2 = \frac{m + n_1^2 n_2}{n_1} \in E_1 \subset C_1$. Let $x := \frac{m}{n_1}$. Then

$x \cdot (n_1 n_2) \in C_1$ and $x + n_1 n_2 \in C_1$

Thm: $\exists\ d : \mathcal{P}(\mathbb{Q}) \to [0,1]$ s.t. $d(\mathbb{Q}) = 1$, d is sub-additive and

$d(A - n) = d(A)$ $\forall A, \forall n \in \mathbb{Q}$ and $d(A/n) = d(A)$. $\frac{1}{n}$

Thm: Let $A \subset \mathbb{Q}$ with $d(A) > 0$. Then

$$d\left(\left\{x : d\left(\{y : \{x+y, xy\} \subset A\}\right) > 0\right\}\right) > 0.$$

$$d(A) = \limsup_{N \to \infty} \frac{|A \cap F_N|}{|F_N|} \quad \text{where} \quad F_N = \left\{ \frac{x}{y} : p_1 \cdots p_N \mid y,\ y \mid p_1^N \cdots p_N^N,\ x \mid p_1^{2N} \cdots p_N^{2N},\ (p_1 \cdots p_N)^N \leq x \leq 4(p_1 \cdots p_N)^N \right\}$$