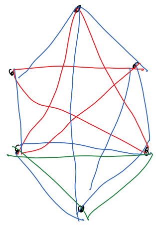
Ramsey's Thm:

$\forall r,k \in \mathbb{N}\ \exists n$ s.t. if $|V|=n$

If $\binom{V}{2} = C_1 \cup \dots \cup C_r$

then $\exists i \in \{1,\dots,r\}, \exists A \subset V$

$|A|=k$ s.t. $\binom{A}{2} \subseteq C_i$



Given a set X and $m \in \mathbb{N}$, denote by

$$\binom{X}{m} = \{A \subset X : |A| = m\}.$$

Infinite Ramsey thm: $\forall r \in \mathbb{N}$, If $\binom{\mathbb{N}}{2} = C_1 \cup \dots \cup C_r$ then $\exists i \in \{1,\dots,r\}\ \exists A \subset \mathbb{N}, |A| = \infty$ s.t. $\binom{A}{2} \subseteq C_i$

Corollary: If $\mathbb{N} = C_1 \cup \dots \cup C_r$, $\exists i \in \{1,\dots,r\}$ and $B \subset \mathbb{N}$ $|B| = \infty$, s.t. $B \oplus B \subset C_i$.

where $B \oplus B = \{b_1 + b_2 : b_1, b_2 \in B,\ b_1 \neq b_2\}$

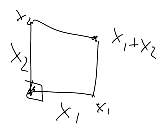
Exercise: Find a 3-coloring of $\mathbb{N}$ without a monoch $B+B$ $(|B| = \infty)$.

Pf of Corollary: Put $\tilde{C}_i = \{\{a,b\} \in \binom{\mathbb{N}}{2} : a+b \in C_i\}$. Let $A \subset \mathbb{N}, |A| = \infty$ s.t. $\binom{A}{2} \subseteq \tilde{C}_i$. Then $A \oplus A \subseteq C_i$

Ramsey thm for hypergraphs: $\forall r\ \forall m$, If $\binom{\mathbb{N}}{m} = C_1 \cup \dots \cup C_r$, $\exists A \subset \mathbb{N}, |A| = \infty$ s.t. $\binom{A}{m} \subseteq C_i$ for some i.

Corollary: If $\mathbb{N} = C_1 \cup \dots \cup C_r$, $\exists A \subset \mathbb{N}$, $|A| = \infty$ s.t. some $C_i \supset A^{\oplus m}$, where

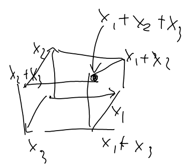
$$A^{\oplus m} = \{a_1 + \dots + a_m;\ a_1, \dots, a_m \in A,\ a_i \neq a_j\ \forall i,j\}$$



Def: An IP-set is a set of the form $\{\sum_{n \in F} n \mid F \subset I, 0 < |F| < \infty\}$ for some infinite $I \subset \mathbb{N}$.

$$= I \cup (I \oplus I) \cup I^{\oplus 3} \cup \dots$$

Thm (Hindman) If $\mathbb{N} = C_1 \cup \dots \cup C_r$, one of the C_i contains an IP-set.



Exercise: Hindman's thm is equivalent to the statement that if one finitely partitions an IP-set, one of the cells of the partitions contains an IP-set. [Hint: $\mathbb{N} = I \cup (I \oplus I) \cup \dots$ for $I = \{2^n : n \in \mathbb{N}\}$]

Def: An idempotent ultrafilter is a collection p of subsets of $\mathbb{N}$ satisfying

1) $\mathbb{N} \in p$, $\emptyset \notin p$
2) If $A \in p$ and $B \supset A$, then $B \in p$

filter

Pf of Hindman's thm: Let p be an idempotent ultrafilter. Let $A \in p$. We will find a sequence $\{x_n\}$ s.t.

A

$A - x_1$

$\{x : A - x \in p\}$

3) If $A, B \in p$ then $A \cap B \in p$

4) If $A \in p$ and $A = A_1 \cup \cdots \cup A_r$, then some $A_i \in p$.

5) $A \in p$ iff $\underbrace{\{n : A - n \in p\}}$ $\in p$

ultrafilter

$\sum_{n \in F} x_n \in A \quad \forall F \subset \mathbb{N}, |F| < \infty$.

choose $x_1 \in A$ s.t. $A - x_1 \in p$.

Note that $\{x : A - x \in p\} \cap A \in p$

choose $x_2 \in A \cap (A - x_1)$ s.t. $A - x_2 \in p$, $A - x_1 - x_2 \in p$

choose $x_3 \in A \cap (A - x_1) \cap (A - x_2) \cap (A - x_1 - x_2)$ s.t.

$A - x_3 \in p$, $A - x_1 - x_3 \in p$, $A - x_2 - x_3 \in p$, $A - x_1 - x_2 - x_3 \in p$

$\{x : \underbrace{A - x_1} - x \in p\}$

Examples: Let $A \subset \mathbb{N}$ and $p = \{B \subset \mathbb{N},\ B \supset A\}$ then p is a filter.

Let $n \in \mathbb{N}$ and $p = \{A \subset \mathbb{N} : n \in A\}$. Then p is an ultrafilter [in fact a principal ultrafilter].

Lemma: There are idempotent ultrafilters. [In fact for any IP-set A, $\exists$ idempotent ultrafilter p s.t. $A \in p$]

Exercise: Let A be an IP-set and $k \in \mathbb{N}$. Then $\exists$ a multiple of k in A.

Question (Erdős): If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, is there $t \in \mathbb{N}$ s.t. $A - t$ contains an IP-set?

Exercise: $\forall \varepsilon > 0\ \exists A \subset \mathbb{N}$ with $d(A) > 1 - \varepsilon$ s.t. $\forall t \in \mathbb{N}\ \exists k \in \mathbb{N}$ s.t. $A - t$ has no multiples of k.

Conjecture (Erdős): If $A \subset \mathbb{N}$ has $d(A) > 0$, $\exists t \in \mathbb{N}$ s.t. $\underbrace{A - t \supset B \oplus B}$. [still open]

infinite.

Ex: If true, one can choose $t \in \{0, 1\}$.

$A \supset B \oplus \underbrace{B + t}_{C}$

Conjecture (Erdős) If $A \subset \mathbb{N}$ has $d(A) > 0$, then $\exists$ infinite sets $B, C \subseteq \mathbb{N}$ s.t. $A \supset B + C$.

Thm (M.-Richter-Robertson): This is true.

Recall correspondence principle: $(\mathbb{N}, d, T : x \mapsto x+1)$ behaves like a m.p.s.

$$\exists B, C \subseteq \mathbb{N} \text{ infinite s.t. } A \supset B + C \iff \exists B \subset \mathbb{N} \text{ infinite s.t. } \Big| \bigcap_{b \in B} (A - b) \Big| = \infty$$

$$\iff \exists B \subset \mathbb{N} \text{ infinite s.t. } \Big| \bigcap_{b \in B} T^{-b} A \Big| = \infty$$

Ex: Consider the doubling map $T : x \mapsto 2x \bmod 1$ on $[0,1]$. Let $A = [0, \frac{1}{2}]$. Then for any infinite set $B \subset \mathbb{N}$, $\mu\left(\bigcap_{b \in B} T^{-b} A\right) = 0$.

(*) Same is true for any Borel set $A \subset [0,1]$, $\mu(A) < 1$.

Proposition: Let $A \subset \mathbb{N}$. Then $\exists$ a topological dynamical system (X, T) and a T-invariant Borel prob. measure μ on X and a point $x \in X$ with a dense orbit and $E \subset X$ s.t. $\mu(E) = d(A)$, E is clopen, and $A = \{n : T^n x \in E\}$.

Fact: If A has $d(A) > 0$, then $A \supset B + C$ where $|B| = 2$ and $d(C) > 0$.

Pf: Let $n \in \mathbb{N}$ s.t. $d(\underbrace{A \cap (A - n)}_{C}) > 0$, let $B = \{0, n\}$, $B + C \subset A$.

In fact, $\forall k \in \mathbb{N}\ \exists B_k, C_k$ s.t. $|B_k| = k$, $d(C_k) > 0$ and $A \supset B_k + C_k$.

Lemma (Bergelson): If $d(A) > 0$, $\exists L \subset \mathbb{N}$, $d(L) \geq d(A)$ and s.t. $\forall B \subset L$, $|B| < \infty$, $\exists C \subseteq \mathbb{N}$ $d(C) > 0$ s.t. $B + C \subseteq A$.

$A_n = A - n$ $\quad (\mathbb{N}, d)$

Lemma: Let (X, μ) be a prob. space. Let $(A_n)_{n \in \mathbb{N}}$ be sets in X s.t. $\mu(A_n) \geq a > 0$. Then $\exists L \subset \mathbb{N}$ s.t. $d(L) \geq a$ and $\forall B \subset L$, $|B| < \infty$, $\mu\left(\bigcap_{b \in B} A_b\right) > 0$.

Pf: Let $f_N = \frac{1}{N} \sum_{n=1}^{N} 1_{A_n}$. Note $\int_X f_N \, d\mu = \frac{1}{N} \sum_{n=1}^{N} \mu(A_n) \geq a$. Therefore $\int_X \limsup_{N \to \infty} f_N \, d\mu \geq a$. For each $x \in X$, let $L_x = \{n : x \in A_n\}$. Then $\bar{d}(L_x) = \limsup_{N \to \infty} f_N(x)$. $\exists x \in X$ s.t. $\bar{d}(L_x) \geq a$. For any $B \subset L$ finite, $\bigcap_{b \in B} A_b \ni \boxed{x}$. Let $\mathcal{F} = \left\{ \bigcap_{n \in F} A_n \;\middle|\; F \subset \mathbb{N},\ |F| < \infty \text{ and } \mu\left(\bigcap_{n \in F} A_n\right) = 0 \right\}$.
Remove from X. $\bigcup_{D \in \mathcal{F}} D$, so that $\bigcap_{b \in B} A_b$ must have positive measure.

Recall: In a m.p.s. $(X, \mathcal{B}, \mu, T)$, a function $f \in L^2$ is weak-mixing if $\forall g \in L^2$,
$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_X T^n f \cdot g \, d\mu \right| = 0$$

Say a set $A \in \mathcal{B}$ is weak mixing if $1_A - \mu(A)$ is weak mixing.

Lemma: If A is weak mixing and has $\mu(A) > 0$, then $\forall B \in \mathcal{B}$ with $\mu(B) > 0$,
$$d\left(\{n : \mu(T^{-n} A \cap B) > 0\}\right) = 1$$

Pf: Exercise.

A set $A \subset \mathbb{N}$ is weak mixing, if it is weak mixing w.r.t. the "m.p.s." $(\mathbb{N}, d, +1)$.

Thm: If $A \subseteq \mathbb{N}$ is weak-mixing, then it contains $B + C$ where B and C are both infinite.

Pf: Let $b_1 \in \mathbb{N}$
$$d\left(\{n : d(\underbrace{(A - b_1)}_{B} \cap \underbrace{(A - n)}_{T^{-n}A}) > 0\}\right) = 1$$

Let $c_1 \in A - b_1$

Let $b_2 \in A - c_1$ s.t. $d((A - b_1) \cap (A - b_2)) > 0$.

Let $c_2 \in (A - b_1) \cap (A - b_2)$ s.t. $d((A - c_1) \cap (A - c_2)) > 0$.

Let $b_3 \in (A - c_1) \cap (A - c_2)$ s.t. $d(\underbrace{(A - b_1) \cap (A - b_2)}_{B} \cap \underbrace{(A - b_3)}_{T^{-b_3}A}) > 0$

$\vdots$

Exercise: A is weak mixing $\Rightarrow A \supset B \oplus B$.