ERGODIC RAMSEY THEORY – EXERCISES WEEK 1

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Exercise 1.3. *Prove that Schur's theorem (Theorem 1.1 in the notes) is equivalent to the following statement:*

"For every $r \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that whenever the set $\{1, \ldots, N\}$ is colored with r colors there is a monochromatic triple of the form $\{x, y, x+y\} \subset \{1, \ldots, N\}$."

Exercise 1.5. *Show that van der Waerden's theorem (Theorem 1.4 in the notes) is equivalent to the following statement:*

"For any $r, k \in \mathbb{N}$ there exists N such that for any coloring of the set $\{1, \ldots, N\}$ with r colors there exists a monochromatic arithmetic progression of the form $\{x, x+y, x+2y, \ldots, x+ky\} \subset \{1, \ldots, N\}$."

Exercise 1.6. *Show that van der Waerden's theorem is equivalent to the following statement:*

"For any finite coloring of $\mathbb{N}$ and for any finite set $F \subset \mathbb{N}$ there exists a monochromatic affine image of F, i.e. there exist $a, b \in \mathbb{N}$ such that the set $aF + b := \{ax + b : x \in F\}$ is monochromatic."

Exercise 1.8. *Show that upper density (defined in Definition 1.7 in the notes) is subadditive and shift invariant, i.e. if $A, B \subset \mathbb{N}$ and $n \in \mathbb{N}$ then $\bar{d}(A \cup B) \leq \bar{d}(A) + \bar{d}(B)$, and $\bar{d}(A - n) = \bar{d}(A)$, where $A - n := \{x \in \mathbb{N} : x + n \in A\}$.*

Exercise 1.10. *Show that Szemerédi's theorem (Theorem 1.9 in the notes) is equivalent to the following statement:*

"For any $\delta > 0$ and $k \in \mathbb{N}$ there exists N such that any set $A \subset \{1, \ldots, N\}$ with $|A| > \delta N$ contains an arithmetic progression of the form $\{x, x+y, x+2y, \ldots, x+ky\}$."

Exercise 1.11. *Let $k \in \mathbb{N}$. Show that there exists $\delta < 1$ such that any set $A \subset \mathbb{N}$ with $\bar{d}(A) > \delta$ contains an arithmetic progression of the form $\{x, x+y, x+2y, \ldots, x+ky\}$.*

Exercise 1.15. *Show that the following are intersective sets (defined in Definition 1.14 in the notes)*

- *Any set with lower density* 1.
- *The set $k\mathbb{N}$ of all multiples of k, for an arbitrary $k \in \mathbb{N}$.*
- $(*)$ *Any set of differences $I - I$ for any infinite set I (not necessarily with positive upper density).*

Exercise 1.16. *Show that the following are not intersective sets.*

- *The odd numbers.*
- *The set $\mathbb{N} \setminus (k\mathbb{N})$ of numbers not divisible by k, for an arbitrary $k \in \mathbb{N}$.*

Exercise 2.2. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be probability spaces and let $T : X \to Y$ be a measurable map.*

- *Show that T preserves the measure (defined in Definition 2.1 in the notes) if and only if for every $f \in L^2(Y)$, the function $f \circ T$ belongs to $L^2(X)$ and satisfies*
$$\int_X f \circ T \, \mathsf{d}\mu = \int_Y f \, \mathsf{d}\nu.$$
- *If both μ and ν are Radon measures, show that T preserves the measure if and only if* (**??**) *holds for every $f \in C(Y)$. [Hint: $C(Y)$ is dense in $L^2(Y)$.]*

Exercise 2.6. *Show that the doubling map (described in Example 2.5 in the notes) does indeed preserve the Lebesgue measure. [Hint: use Exercise 2.2]*

Exercise 2.9. *Show that the set $2\mathbb{N}$ of even numbers is a set of recurrence (defined in Definition 2.8 in the notes) but the set $2\mathbb{N} - 1$ of odd numbers is not.*

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Exercise 2.12. (∗) *Show that if $R \subset \mathbb{N}$ is a set of recurrence and is decomposed as $R = A \cup B$ then either A or B is a set of recurrence. [Hint: Proceed by contradiction and take the product system of the two presumed counter-examples.]*

Exercise 2.19. *Show that*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} h_n = c$$

is equivalent to

$$\forall\ (I_N)_{N\in\mathbb{N}} \qquad \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} h_n = c$$

where $(I_N)_{N\in\mathbb{N}}$ is a sequence of intervals $I_N = \{a_N + 1, a_N + 2, \ldots, a_N + b_N\}$ whose lengths b_N tend to infinity.

Exercise 2.23. *Let (a_n) be a sequence of non-negative real numbers and let $a \in \mathbb{R}$. Show that if*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} a_n = a,$$

then for every $\epsilon > 0$ the set

$$\{n \in \mathbb{N} : a_n \geq a - \epsilon\}$$

is syndetic (defined in Definition 2.22 in the notes).