ERGODIC RAMSEY THEORY – EXERCISES WEEK 2

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Exercise 3.1. *Show that there are sets* $A, B \subset \mathbb{N}$ *with* $\bar{d}(A) = \bar{d}(B) = 1$ *but* $A \cap B = \emptyset$*.*

Exercise 3.2. *Show that there are sets* $A, B \subset \mathbb{N}$ *both having natural density but such that* $A \cap B$ *does not.*

Exercise 3.5. *Show that the measure* μ *constructed in the proof of the Furstenberg Correspondence Principle (Theorem 3.4 in the notes) is* T*-invariant. [Hint: Using Exercise 2.2, it suffices to show that* $\int_X f \, \mathsf{d}\mu = \int_X f \circ T \, \mathsf{d}\mu$ *for every* $f \in C(X)$*.]*

Exercise 3.6. *Adapting the proof of Proposition 2.11 in the notes, show that Szemerédi's theorem (Theorem 1.9 in the notes) implies Furstenberg's Multiple Recurrence Theorem (Theorem 3.3 in the notes).*

Exercise 3.7. *Show that, in the proof of Proposition 2.11 in the notes, the function* $x \mapsto \bar{d}(E_x)$ *is measurable and hence we can in fact consider its integral.*

Exercise 3.8. *Show that, in the proof of Proposition 2.11 in the notes, for* μ*-a.e.* $x \in X$ *the set* E_x *has a natural density, i.e., show that the limit* $\lim_{N\to\infty} \frac{1}{N} |E_x \cap \{1, \ldots, N\}|$ *exists.*

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