ERGODIC RAMSEY THEORY – EXERCISES WEEK 3

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Exercise 4.4. *Show that the sequence* $x_n = \sqrt{n} \bmod 1$ *is uniformly distributed.*

Exercise 4.5. *Show that the sequence* $x_n = \log n \bmod 1$ *is* **not** *uniformly distributed.*

Exercise 4.9. *Adapt the proof of Lemma 4.8 to the following version for uniform Cesàro averages (see Remark 2.18): Let* H *be a Hilbert space and let* $(x_n)_{n=1}^{\infty}$ *be a bounded sequence taking values in* H*. If for every* $d \in \mathbb{N}$*,*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \langle x_{n+d}, x_n \rangle = 0 \tag{4.1}$$

then

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} x_n = 0.$$

Exercise 4.10. $(*)$

Let $p \in \mathbb{R}[x]$ *have at least one irrational coefficient (other than the constant term) and let* $U \subset [0,1]$ *be open and non-empty. Is it true that the set* $\{n \in \mathbb{N} : p(n) \bmod 1 \in U\}$ *is syndetic? [Hint: Use Exercise 4.9 to obtain versions of Lemma 4.6 and Corollary 4.7 for uniform Cesàro averages and then use a similar argument as for Exercise 2.23.]*

Exercise 4.20. *Finish the proof of Theorem 4.19 by explicitly describing the measure preserving system structure of* Y *and showing that* π *is indeed a factor map.*

Exercise 4.21. *Let* $(X, \mathcal{A}, \mu, T)$ *be a measure preserving system and let* $(Y, \mathcal{B}, \nu, S)$ *be a factor. Prove that:*

- *If* $(X, \mathcal{A}, \mu, T)$ *is ergodic, then so is* $(Y, \mathcal{B}, \nu, S)$*.*
- *If* $(X, \mathcal{A}, \mu, T)$ *is totally ergodic, then so is* $(Y, \mathcal{B}, \nu, S)$*.*

Exercise 4.22. *Show that if* $f, g \in H_{rat}$ *are bounded, then their product* $f \cdot g$ *is also in* H_{rat}*. Can you find an example showing that the same is not tru for* H_{te}*?*

Exercise 4.23. $(*)$

Show that the collection $\{A \in \mathcal{B} : 1_A \in H_{rat}\}$ *is a* σ*-algebra.*

Exercise 4.27. *Adapt the proof of Theorem 4.26 to obtain that, under the same conditions, if additionally* $\mu(A) > 0$*, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-p(n)} A) > 0.$$

Exercise 4.28. $(*)$ *Using Exercise 4.9 in the proof of Theorem 4.26, show that for any set* $E \subset \mathbb{N}$ *with* $\bar{d}(E) > 0$*, the set* $\{n \in \mathbb{N} : n^2 \in E - E\}$ *is syndetic.*

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