# ERGODIC RAMSEY THEORY – EXERCISES WEEK 4

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**Exercise 5.2.** *Show that in Theorem 3.4 one can obtain an invertible system. In other words, show that for any $E \subset \mathbb{N}$ there exist an* **invertible** *measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(E)$ such that for any $n_1, \dots, n_k \in \mathbb{N}$,*

$$\mu\big(A \cap T^{-n_1} A \cap \cdots \cap T^{-n_k} A\big) \leq \bar{d}\big(E \cap (E - n_1) \cap \cdots \cap (E - n_k)\big).$$

*[Hint: Think of $E$ as a subset of $\mathbb{Z}$, and replace everywhere in the proof $\mathbb{N}$ with $\mathbb{Z}$.]*

**Exercise 5.6.** *Using Theorem 5.3 and the simplifications made at the beginning of this subsection, show that in Theorem 5.1 we can assume that the system is ergodic (in other words, show that if we Theorem 5.1 holds for ergodic systems then it holds for any measure preserving system).*

**Exercise 5.13.** *Show that the doubling map $x \mapsto 2x \bmod 1$ on $[0, 1)$ with respect to the Lebesgue measure is a weak-mixing system.*

**Exercise 5.16.** *Show that $H_{wm}$ is a closed $T$-invariant subspace of $L^2$.*

**Exercise 5.20.** $(*)$ *Let $(X\ B, \mu, T)$ be a measure preserving system and let $f \in L^2$ be such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \big|\langle T^k f, f\rangle\big| = 0.$$

*Prove that $f$ is weak mixing. [Hint: Using Lemma 5.18 with $u_n = \langle T^n f, g\rangle T^n f$.]*

**Exercise 5.22.** *Show that in the system $(X, \mathcal{B}, \mu, T)$ where $X = [0, 1)$, $\mu$ is the Lebesgue measure and $T : x \mapsto x + \alpha \bmod 1$ for some irrational $\alpha$, every $f \in L^2$ is compact.*

**Exercise 5.23.** *Show that if $f$ is compact, then for every $\epsilon > 0$ the set $\{n \in \mathbb{N} : \|T^n f - f\| < \epsilon\}$ is syndetic.*

**Exercise 5.25.** $(*)$ *Show directly from the definition that $H_c$ is a closed $T$-invariant subspace of $L^2$.*

**Exercise 5.26.** *Let $X = \mathbb{T}^2$ have the Borel $\sigma$-algebra and the Haar measure and let $T : (x, y) \mapsto (x+\alpha, y+x)$ for some fixed irrational $\alpha$.*

(1) *Show that every function of the form $f(x, y) = e^{2\pi i n x}$ with $n \in \mathbb{Z}$ is compact.*
(2) *Show that every function of the form $f(x, y) = e^{2\pi i(nx+my)}$, with $(n, m) \in \mathbb{Z}^2$ and $m \neq 0$, is weak mixing.*
(3) *Show that the conclusion of Theorem 5.24 holds in this system (without using the theorem) by explicitly describing $H_c$ and $H_{wm}$.*

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