ERGODIC RAMSEY THEORY – EXERCISES WEEK 6

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Exercise 7.3. *Show that, using only Szemerédi's theorem, one can deduce that any subset of* $\mathbb{N}^2$ *with positive upper density contains a rectangular* $k \times k$ *grid, i.e. a set of the form*

$$\big\{(x_1, x_2) + (in, jm) : 1 \le i, j \le k\big\}$$

for some $x_1, x_2, n, m \in \mathbb{N}$.

Exercise 7.6. *Adapt the proof of Theorem 3.4 to give a proof of Proposition 7.4. [Hint: Take* $X = \{0,1\}^{N_0^d}$, *let* T_i *be the shift in the* i*-th direction and let* $A = \{x \in X : x_{(0,\dots,0)} = 1\}$*.]*

Exercise 7.7. *Show that Theorem 7.1 follows from combining Theorem 7.2 with Proposition 7.4.*

Exercise 8.4. *Show that a system* (X, T) *is minimal if and only if every point* $x \in X$ *has a dense orbit (the orbit of a point* $x \in X$ *is the set* $\{T^n x : n \in \mathbb{N}\}$*).*

Exercise 8.5. *Show that if a system* (X, T) *is minimal then* $T : X \to X$ *is surjective.*

Exercise 8.9. *Using Proposition 8.6, show that Theorems 1.4 and 8.7 are equivalent.*

[Hint: To show that Theorem 1.4 implies Theorem 8.7, take any point x *in* X *and construct a coloring of* $\mathbb{N}$ *by looking at the orbit of* x*.]*

Exercise 8.10. *Show that in Theorem 8.7, the assumption that* (X, T) *is minimal is not needed.*

Exercise 8.15. *Prove that the following are all equivalent statements:*

(1) *Theorem 8.11.*
(2) *Theorem 8.12.*
(3) *Theorem 8.13.*
(4) *Theorem 8.12 without the minimality assumption.*

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