ERGODIC RAMSEY THEORY – EXERCISES WEEK 8

JOEL MOREIRA

Exercise 9.3. *Show that Theorem 9.3 implies Theorem 9.1.*

Exercise 9.5. *Find a* 3*-coloring of* $\mathbb{N}$ *without a monochromatic set of the form* $I+I=\{x+y : x,y\in I\}$*. [Hint: Color* $\mathbb{N}$ *with longer and longer intervals of alternating colors to avoid any pair* $\{x,2x\}$*. By using the third color one can avoid pairs* $\{x+y,2x\}$ *when* $y\ll x$*.]*

Exercise 9.10. *Show that Theorem 9.9 is equivalent to the statement that any finite coloring of an IP-set yields a monochromatic IP-set. [Hint:* $\mathbb{N}$ *is the IP-set generated by the set* $I=\{2^n : n\in\mathbb{N}\}$*.]*

Exercise 9.13. *Let* $k\in\mathbb{N}$ *and let* A *be an IP-set. Show that* A *contains a multiple of* k*.*

Exercise 9.15. *Show that for every* $\epsilon>0$ *there exists a set* $A\subset\mathbb{N}$ *with upper density* $\bar{d}(A)>1-\epsilon$ *and such that for any* $t\in\mathbb{N}$ *there exists* $k=k(t)$ *such that* $A-t$ *has no multiples of* k*.*

Exercise 9.17. *Find a set* $A\subset\mathbb{N}$ *with* $\bar{d}(A)>0$ *and such that for any* t *and any infinite set* $B\subset\mathbb{N}$*,* $A-t$ *does not contain* $B+B$*, and hence a restricted sum* $B\oplus B$ *is required in Conjecture 9.16.*

Exercise 9.18. *Show that if Conjecture 9.16 holds, then one can always take* $t\in\{0,1\}$*.*

Exercise 9.20. *Show that Conjecture 9.16 implies Conjecture 9.19.*

Exercise 9.21. *Consider the doubling map (i.e. the transformation* $x\mapsto 2x \bmod 1$ *on* $[0,1)$ *with the Lebesgue measure) and let* $A=[0,1/2)$*. Show that for any infinite set* $B\subset\mathbb{N}$*, the intersection* $\bigcap T^{-b}A$ *has zero measure.*

Exercise 9.22. $(*)$ *Consider the doubling map (i.e. the transformation* $x\mapsto 2x \bmod 1$ *on* $[0,1)$ *with the Lebesgue measure) and let* $A\subset[0,1)$ *be any Borel set. Show that for any infinite set* $B\subset\mathbb{N}$*, the intersection* $\bigcap T^{-b}A$ *has zero measure.*

Exercise 9.25. *Let* $(X,\mathcal{B},\mu,T)$ *be a m.p.s. and let* $A\in\mathcal{B}$ *be such that* $\mu(A)>0$ *and the function* $1_A-\mu(A)$ *is weak mixing. Show that for any* $B\in\mathcal{B}$ *with* $\mu(B)>0$*, the set*

$$R:=\{n\in\mathbb{N} : \mu(A\cap T^{-n}B)>0\}$$

has full natural density, i.e. $d(R)=1$ *(which is stronger than just* $\bar{d}(R)=1$*).*

Exercise 9.26. *Show that any weak-mixing set* $A\subset\mathbb{N}$ *with positive upper density contains* $B\otimes B$ *for some infinite set* $B\subset\mathbb{N}$*.*

Date: December 8, 2021.