7. Extensions of Szemerédi's theorem

Shortly after Furstenberg published his ergodic theoretic proof of Szemerédi's theorem, in joint work with Katznelson they established a multidimensional version. For many years, the only known proofs of this multidimensional Szemerédi theorem (Theorem 7.1 below) involved ergodic theory.

Let $d \in \mathbb{N}$. Given a set $A \subset \mathbb{N}^d$, its upper density is defined by

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{1}{N^d}\big|A \cap \{1,\dots,N\}^d\big|.$$

Theorem 7.1 (Furstenberg-Katznelson [10])**.** *If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then for every finite set $F \subset \mathbb{N}^d$ there exists $n \in \mathbb{N}$ and $x \in \mathbb{N}^d$ such that*

$$A \supset x + nF := \{x + nv : v \in F\}.$$

For instance, if $d = 2$ and $F = \{0,1,\dots,k\}^2$, it follows from Theorem 7.1 that any subset of $\mathbb{N}^2$ with positive upper density contains a square $k \times k$ grid.

Exercise 7.2. *Show that, using only Szemerédi's theorem, one can deduce that any subset of $\mathbb{N}^2$ with positive upper density contains a rectangular $k \times k$ grid, i.e. a set of the form*

$$\big\{(x_1, x_2) + (in, jm) : 1 \le i, j \le k\big\}$$

for some $x_1, x_2, n, m \in \mathbb{N}$.

Here's the multiple recurrence theorem they established.

Theorem 7.3. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T_1, \dots, T_d : X \to X$ be commuting measure preserving transformations. Then for any $A \subset \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that*

$$\mu\big(A \cap T_1^{-n}A \cap \dots \cap T_d^{-n}A\big) > 0.$$

Exercise 7.4. *Show that Theorem 7.3 implies that, under the same conditions, for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that*

$$\mu\left(A \cap \bigcap_{i=1}^{d}\bigcap_{j=1}^{k} T_i^{-jn}A\right) > 0.$$

To show that Theorem 7.3 implies Theorem 7.1, one needs a suitable extension of the Correspondence Principle.

Proposition 7.5. *Let $d \in \mathbb{N}$ and $E \subset \mathbb{N}^d$. Then there exists a probability space $(X, \mathcal{B}, \mu)$, commuting measure preserving transformations $T_1, \dots, T_d$ on X and a set $A \in \mathcal{B}$ such that $\mu(A) = \bar{d}(E)$ and for any $n_1, \dots, n_k \in \mathbb{N}^d$, say $n_i = (n_{i,1}, \dots, n_{i,d})$, we have*

$$\bar{d}\big(E \cap (E - n_1) \cap \dots \cap (E - n_k)\big) \ge \mu\left(A \cap \bigcap_{i=1}^{k} T_1^{-n_{i,1}} T_2^{-n_{i,2}} \cdots T_d^{-n_{i,d}} A\right)$$

Exercise 7.6. *Adapt the proof of Theorem 3.4 to give a proof of Proposition 7.5. [Hint: Take $X = \{0,1\}^{N_0^d}$, let T_i be the shift in the i-th direction and let $A = \{x \in X : x_{(0,\dots,0)} = 1\}$.]*

Exercise 7.7. *Show that Theorem 7.1 follows from combining Theorem 7.3 with Proposition 7.5.*

The proof of Theorem 7.3 follows the same basic structure as the proof of Theorem 3.3. In particular, it uses the idea of exhausting the system $(X, \mathcal{B}, \mu, T_1, \dots, T_d)$ by weak mixing and compact extensions; although in this situation one also needs to consider more general behaviour.

Later Bergelson and Leibman proved the polynomial version of Szemerédi's theorem, Theorem 1.18. In fact they proved a multidimensional version as well. The polynomial Szemerédi theorem is deduced (using the Correspondence Principle) from the following polynomial multiple recurrence result:

Theorem 7.8. *Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system and let $p_1, \dots, p_k \in \mathbb{Z}[x]$ satisfy $p_i(0) = 0$. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that*

$$\mu\big(A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A\big) > 0.$$

The proof of Theorem 7.8 follows the strategy implemented by Furstenberg, and in particular uses directly Theorem 6.19 and analogues of Theorems 6.17 and 6.18. To lift the polynomial recurrence property over weak-mixing extensions one can use the van der Corput trick and a similar argument to the linear case. However, in order to lift the polynomial recurrence property over compact extensions, one requires a suitable version of the van der Warden theorem (Theorem 1.4).

Theorem 7.9. *Let $p_1, \dots, p_k \in \mathbb{Z}[x]$ satisfy $p_i(0) = 0$. For any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$ of $\mathbb{N}$ there exists $x, n \in \mathbb{N}$ and $C \in \{C_1, \cdots, C_r\}$ such that*

$$\{x, x + p_1(n), \cdots, x + p_k(n)\} \subset C.$$

It is clear the Theorem 7.9 is a corollary of Theorem 1.18; however it is required to prove Theorem 1.18, so one needs to be able to prove Theorem 7.9 directly.

8. Coloring theorems and Topological Dynamics

It turns out that to prove coloring results such as van der Waerden's theorem, ergodic theory isn't as suitable as another branch of dynamics, called topological dynamics.

Definition 8.1. *A* **topological dynamical system** *(or simply* **system***) is a pair (X, T) where X is a compact metric space and $T : X \to X$ is continuous.*

Given a system (X, T), any closed set $Y \subset X$ satisfying $TY \subset Y$ gives rise to a subsystem (Y, T).

Definition 8.2. *A system (X, T) is* **minimal** *if there is no proper subsystem.*

An application of Zorn's lemma shows that any topological dynamical system has a minimal subsystem.

Exercise 8.3. *Show that a system (X, T) is minimal if and only if every point $x \in X$ has a dense orbit (the orbit of a point $x \in X$ is the set $\{T^n x : n \in \mathbb{N}\}$).*

Exercise 8.4. *Show that if a system (X, T) is minimal then $T : X \to X$ is surjective.*

Proposition 8.5. *If (X, T) is minimal and $A \subset X$ is open and non-empty, then there exists $n \in \mathbb{N}$ such that $A \cap T^{-n} A \neq \emptyset$.*

Proof. The set $B := X \setminus \bigcup_{n \in \mathbb{N}} T^{-n} A$ is closed and $TB \subset B$. Since $A \neq \emptyset$, $B \neq X$, and hence by minimality $B = \emptyset$. It follows that $\bigcup_{n \in \mathbb{N}} T^{-n} A = X$ and hence some $T^{-n} A$ must have non-empty intersection with A. □

The connection between coloring theorems and topological dynamics is given by the following instance of the correspondence principle.

Proposition 8.6. *Let $\mathbb{N} = C_1 \cup \cdots \cup C_r$ be an arbitrary finite coloring of $\mathbb{N}$. There exists a minimal topological dynamical system (X, T) and a cover $X = A_1 \cup \cdots \cup A_r$ by open sets such that for any $n_1, \dots, n_k \in \mathbb{N}$ and any $i \in \{1, \dots, r\}$,*

$$A_i \cap T^{-n_1} A_i \cap \cdots \cap T^{-n_k} A_i \neq \emptyset \quad \Rightarrow \quad C_i \cap (C_i - n_1) \cap \cdots \cap (C_i - n_k) \neq \emptyset$$

Proof. Let $X_0 = \{1, \dots, r\}^{\mathbb{N}_0}$, let $T : X_0 \to X_0$ be the left shift and let $\chi \in X_0$ be the function $\chi = \sum i 1_{C_i}$. Let $X_1 = \overline{T^n \chi : n \in \mathbb{N}}$ be the orbit closure of χ, notice that (X_1, T) is a subsystem of (X_0, T), and let $X \subset X_1$ be a minimal subsystem.

Let $A_i := \{x \in X : x_0 = i\}$. If $y \in A_i \cap T^{-n_1} A_i \cap \cdots \cap T^{-n_k} A_i$ for some i and $n_1, \dots, n_k$, then $y_0 = y_{n_1} = \cdots = y_{n_k} = i$. Since $y \in X \subset X_1$, there exists a point $T^n \chi$ in the orbit of χ such that $(T^n \chi)_m = y_m$ for every $m \leq n_k$. In particular $(T^n \chi)_{n_j} = i$ for every $j = 0, \dots, k$ (where for convenience we define $n_0 = 0$) which means that $\chi_{n+n_j} = i$ for every j and hence that $n \in C_i \cap (C_i - n_1) \cap \cdots \cap (C_i - n_k)$. □

In view of Proposition 8.6, van der Warden's theorem follows from the following multiple recurrence theorem.

Theorem 8.7. *Let (X, T) be a minimal system and $X = C_1 \cup \cdots \cup C_r$ a finite open cover of X. Then for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $i \in \{1, \dots, r\}$ such that*

$$C_i \cap T^{-n} C_i \cap \cdots \cap T^{-kn} C_i \neq \emptyset.$$

Exercise 8.8. *Using Proposition 8.6, show that Theorems 1.4 and 8.7 are equivalent.*

[Hint: To show that Theorem 1.4 implies Theorem 8.7, take any point x in X and construct a coloring of $\mathbb{N}$ by looking at the orbit of x.]

We assume minimality in the statement of Theorem 8.7 because it makes the proof easier (similar to how we assume ergodicity in the proof of Theorem 6.1). However it can be shown directly that this assumption can be discarded.

Exercise 8.9. *Show that in Theorem 8.7, the assumption that (X, T) is minimal is not needed.*

It turns out that Theorem 8.7 is equivalent to a version closer to Theorem 3.3.

Lemma 8.10. *Suppose Theorem 8.7 holds for some $k \in \mathbb{N}$. Then for any minimal system (X, T) and any open $A \subset X$, if $A \neq \emptyset$ then there exists $n \in \mathbb{N}$ such that $A \cap T^{-n}A \cap \cdots \cap T^{-kn}A \neq \emptyset$.*

Proof. As we've seen above, $X = \bigcup_{i\in\mathbb{N}} T^{-i}A$. By compactness it follows that $X = \bigcup_{i=1}^{N} T^{-i}A$ for some $N \in \mathbb{N}$. Using Theorem 8.7 we find $i \leq N$ and $n \in \mathbb{N}$ such that $\emptyset \neq T^{-i}A \cap T^{-n}T^{-i}A \cap \cdots \cap T^{-kn}T^{-i}A = T^{-i}\big(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A\big)$, which implies that $A \cap T^{-n}A \cap \cdots \cap T^{-kn}A \neq \emptyset$. □

Proof of Theorem 8.7. The proof goes by induction over k. The case $k = 1$ follows immediately from Proposition 8.5.

Next, suppose $k > 1$ and the result has been established for any smaller value of k. Some C_i must be non-empty; suppose WLOG $C_1 \neq \emptyset$. Then apply the induction hypothesis and Lemma 8.10 to find $n_1 \in \mathbb{N}$ such that $B_1 := C_1 \cap T^{-n_1}C_1 \cap \cdots \cap T^{-(k-1)n_1}C_1 \neq \emptyset$.

We now consider two cases. In the first case $T^{-n_1}B_1 \cap C_1 \neq \emptyset$. But then $C_1 \cap T^{-n_1}C_1 \cap \cdots \cap T^{-kn_1}C_1 \neq \emptyset$ and we are done.

The second case is when $T^{-n_1}B_1 \cap C_1 = \emptyset$. In this case, $T^{-n_1}B_1$ must have a non-empty intersection with some other C_i; WLOG suppose $D_2 := T^{-n_1}B_1 \cap C_2 \neq \emptyset$. We can now invoke again the induction hypothesis and Lemma 8.10 to find $n_2 \in \mathbb{N}$ such that $B_2 := D_2 \cap T^{-n_2}D_2 \cap \cdots \cap T^{-(k-1)n_2}D_2 \neq \emptyset$. We consider three new subcases.

In the first case $T^{-n_2}B_2 \cap C_2 \neq \emptyset$. But then (since $D_2 \subset C_2$), $C_2 \cap T^{-n_2}C_2 \cap \cdots \cap T^{-kn_2}C_2 \neq \emptyset$ and we are done.

In the second case, $T^{-n_2}B_2 \cap C_1 \neq \emptyset$. But then (since $D_2 \subset T^{-n_1}B_1 \subset T^{-in_1}C_1$ for each $i \in \{1, \dots, k\}$), $C_1 \cap T^{-(n_1+n_2)}C_1 \cap \cdots \cap T^{-k(n_1+n_2)}C_1 \neq \emptyset$ and we are done.

In the third case $T^{-n_2}B_2$ must have a non-empty intersection with some other C_i; WLOG suppose $D_3 := T^{-n_2}B_2 \cap C_3 \neq \emptyset$.

We can continue in this manner, but since we start with a finite open cover, after r steps we do not have a final case and the proof will finish. □

Using a similar strategy, we can establish directly the following coloristic corollary of Sàrközy's theorem.

Theorem 8.11. *If $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists $C \in \{C_1, \dots, C_r\}$ and $n, x \in \mathbb{N}$ such that $\{x, x + n^2\} \subset C$.*

The dynamical version of Theorem 8.11 is the following.

Theorem 8.12. *Let (X, T) be a minimal system and suppose $X = C_1 \cup \cdots \cup C_r$ is an open cover. Then there exists $C \in \{C_1, \dots, C_r\}$ and $n \in \mathbb{N}$ such that $T^{-n^2}C \cap C \neq \emptyset$.*

Similarly to Lemma 8.10, one can write an equivalent formulation of Theorem 8.11 using a single open set.

Theorem 8.13. *Let (X, T) be a minimal system and let $A \subset X$ be open and non-empty. Then for some $n \in \mathbb{N}$, $A \cap T^{-n^2}A \neq \emptyset$.*

Exercise 8.14. *Prove that the following are all equivalent statements:*

1. *Theorem 8.11.*
2. *Theorem 8.12.*
3. *Theorem 8.13.*
4. *Theorem 8.12 without the minimality assumption.*