8.1. **Piecewise syndetic sets.** We've encountered above the notion of syndetic sets: subsets of $\mathbb{N}$ with bounded gaps. The dual notion to syndetic sets is that of thick sets.

Definition 8.15. *A set $T \subset \mathbb{N}$ is* **thick** *if it contains arbitrarily long intervals, i.e.,*

$$\forall N \in \mathbb{N}\ \exists m_N \in \mathbb{N}\ s.t.\ \{m_N, m_N+1, \dots, m_N+N\} \subset T.$$

Exercise 8.16.
(1) *Show that a set $T \subset \mathbb{N}$ is thick if and only if its complement $\mathbb{N} \setminus T$ is not syndetic.*
(2) *Show that a set $S \subset \mathbb{N}$ is syndetic if and only if its complement $\mathbb{N} \setminus S$ is not thick.*
(3) *Show that a set $T \subset \mathbb{N}$ is thick if and only if for any syndetic set $S \subset \mathbb{N}$, the intersection $S \cap T \neq \emptyset$.*
(4) *Show that a set $S \subset \mathbb{N}$ is syndetic if and only if for any thick set $T \subset \mathbb{N}$, the intersection $S \cap T \neq \emptyset$.*

Definition 8.17. *A set $A \subset \mathbb{N}$ is* **piecewise syndetic** *if $A = S \cap T$ for a syndetic set $S \subset \mathbb{N}$ and a thick set $T \subset \mathbb{N}$.*

Note that all three notions of syndetic, thick and piecewise syndetic are upwards closed, i.e. if A possesses one of those properties and $B \supset A$, then B also possesses the same property.

The relation between piecewise syndetic sets and partition Ramsey theory is made apparent by the following lemma.

Lemma 8.18 (Brown's lemma)**.** *Let A be piecewise syndetic, and suppose that $A = A_1 \cup \dots \cup A_r$. Then at least one of the A_i is piecewise syndetic.*

Proof. By an inductive argument it suffices to prove the lemma when $r = 2$. Suppose $A = S \cap T = A_1 \cup A_2$ where S is syndetic and T is thick. Let $\tilde{S} = S \setminus A_2$. If $\tilde{S}$ is syndetic, then $A_1 = \tilde{S} \cap T$ is piecewise syndetic. If $\tilde{S}$ is not syndetic, then its complement $\tilde{T} := \mathbb{N} \setminus \tilde{S}$ is thick, and hence $A_2 = \tilde{T} \cap S$ is piecewise syndetic. □

Since $\mathbb{N}$ is piecewise syndetic, for any coloring of $\mathbb{N}$ one of the colors is piecewise syndetic. Therefore, if one seeks to show that any finite coloring of $\mathbb{N}$ contains a certain monochromatic pattern, it suffices to show that every piecewise syndetic set contains it.

8.2. **Minimal systems and (piecewise) syndetic sets.** Let (X, T) be a topological dynamical system, let $U \subset X$ be open and let $x \in X$. We denote by $V(x, U) := \{n \in \mathbb{N} : T^n x \in U\}$ the set of visit times of x to U. The connection between minimal systems and syndetic sets is given in the following lemma.

Lemma 8.19. *A system (X, T) is minimal if and only if for every non-empty open set $U \subset X$ and every $x \in X$, the set $V(x, U)$ is syndetic.*

Proof. If for every non-empty open set $U \subset X$ and every $x \in X$, the set $V(x, U)$ is syndetic, then in particular $V(x, U) \neq \emptyset$ and it follows that every point has a dense orbit. In view of Exercise 8.3, (X, T) is minimal.

Conversely suppose that (X, T) is minimal and let $U \subset X$ be open and non-empty, and let $x \in X$. Then $Y := X \setminus \bigcup_{i=0}^{\infty} T^{-i}U$ is a closed and T-invariant subset of X which is not all of X since $U \cap Y = \emptyset$. By minimality it follows that $Y = \emptyset$ and hence $X = \bigcup_{i=0}^{\infty} T^{-i}U$. By compactness there exists $r \in \mathbb{N}$ such that $X = \bigcup_{i=0}^{r} T^{-i}U$. Given any $n \in \mathbb{N}$, the point $T^n x \in X$ must belong to one of the $T^{-i}U$ and hence $T^{n+i}x \in U$. In other words, for every $n \in \mathbb{N}$ there exists $i \in \{0, \dots, r\}$ such that $n + i \in V(x, U)$, and this implies that $V(x, U)$ is syndetic. □

A topological dynamical system (X, T) is called ***transitive*** if there exists at least one point with a dense orbit. In a transitive system, we can replace sets of visits with the closely related sets of return times (sets of visits $V(x, U)$ where $x \in U$).

Exercise 8.20. *Show that a transitive system (X, T) is minimal if and only if for every for every non-empty open set $U \subset X$ and every $x \in U$, the set $V(x, U)$ is syndetic.*

There is a version of Lemma 8.19 that applies to non-minimal systems. Recall that every system (X, T) has a minimal subsystem.

Lemma 8.21. *Let (X, T) be a transitive system, suppose $x \in X$ has a dense orbit, let $Y \subset X$ be a minimal subsystem and let $U \subset X$ be an open set such that $U \cap Y \neq \emptyset$. Then $V(x, U)$ is piecewise syndetic.*

Proof. Let $y \in Y$ and let $S = V(y, U)$. By Lemma 8.19, S is syndetic, so there exists $r \in \mathbb{N}$ such that $S - \{1, \dots, r\} = \mathbb{N}$.

For each $N \in \mathbb{N}$ let $m_N \in \mathbb{N}$ be such that $T^{m_N}x$ is so close to y that for each $n \in \{0, 1 \dots, N\}$, whenever $T^n y \in T^{-n}U$, also $T^n(T^{m_N}x) \in T^{-n}U$. Therefore $V(x, U) \supset m_N + \big(S \cap \{0, \dots, N\}\big)$ for every $N \in \mathbb{N}$. We claim that the union $A := \bigcup_{N\in\mathbb{N}} m_N + \big(S \cap \{0, \dots, N\}\big)$ is piecewise syndetic, and this will finish the proof.

Indeed, the union $T = \bigcup_{N\in\mathbb{N}} m_N + \{0, \dots, N\}$ is thick, and letting $\tilde{S} := (\mathbb{N} \setminus T) \cup A$ we clearly have $A = \tilde{S} \cap T$, so it suffices to prove that $\tilde{S}$ is syndetic. Take any $x \in \mathbb{N}$. If $x \notin T$, then $x \in \tilde{S}$. Otherwise, $x \in m_N + \{0, \dots, N\}$ for some $N \in \mathbb{N}$, so that $x = m_N + n$ for some $n \in \{0, \dots, N\}$. We can then find $i \in \{1, \dots, r\}$ such that $n + i \in S$, and hence $x + i = m_N + n + i \in \tilde{S}$. We conclude that $\tilde{S}$ is syndetic and hence A is piecewise syndetic. □

8.3. **Partition regular patterns.** We will use Lemma 8.21 to derive a strengthening of the van der Waerden theorem (Theorem 1.4). In fact, we will develop a more general framework: call a ***pattern on*** $\mathbb{N}$ a collection $\mathcal{P}$ of finite subsets of $\mathbb{N}$. Elements of a pattern $\mathcal{P}$ may be called ***configurations***. The pattern is ***shift invariant*** if for every $C \in \mathcal{P}$ and $n \in \mathbb{N}$ also $C + n \in \mathcal{P}$. We say that the pattern $\mathcal{P}$ is ***monochromatic*** if for every finite coloring of $\mathbb{N}$ there exists $C \in \mathcal{P}$ which is monochromatic.

Example 8.22. *Let $R \subset \mathbb{N}$ and let $\mathcal{P} := \{\{x, x + r\} : x \in \mathbb{N}, r \in R\}$. Then $\mathcal{P}$ is a shift invariant pattern. If R is the set of perfect squares, then $\mathcal{P}$ is monochromatic, in view of Theorem 8.11.*

Theorem 8.23. *Let $\mathcal{P}$ be a shift invariant pattern. Then the following are equivalent:*

1. *$\mathcal{P}$ is monochromatic.*
2. *For every minimal system (X, T) and every non-empty open $U \subset X$, there is a configuration $C \in \mathcal{P}$ such that $\bigcap_{n\in C} T^{-n}U \neq \emptyset$.*
3. *For every topological system (X, T) and every finite open cover $X = C_1 \cup \dots \cup C_r$, there is a configuration $C \in \mathcal{P}$ and $i \in \{1, \dots, r\}$ such that $\bigcap_{n\in C} T^{-n}C_i \neq \emptyset$.*
4. *For every $r \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for any coloring of the interval $[1, N]$ with r colors there exists $C \in \mathcal{P}$ contained in $[1, N]$ which is monochromatic.*
5. *For every syndetic set $S \subset \mathbb{N}$ there exists $C \in \mathcal{P}$ such that $C \subset S$.*
6. *For every piecewise syndetic set $A \subset \mathbb{N}$ there exists $C \in \mathcal{P}$ such that $C \subset A$.*
7. *For every piecewise syndetic set $A \subset \mathbb{N}$ there exists $C \in \mathcal{P}$ such that the set*
$$\big\{n \in \mathbb{N} : C + n \subset A\big\}$$
is piecewise syndetic.

Proof. The implication (3)⇒(1) follows at once from Proposition 8.6. To see why (1)⇒(2), take a minimal system (X, T), an open set $\emptyset \neq U \subset X$ and let $x \in X$ be arbitrary. As we saw before, finitely many pre-images of U cover X (by minimality and compactness) so we can color $n \in \mathbb{N}$ according to which pre-image of U contains the point $T^n x$. Applying (1) to this coloring it follows that (2) holds. The implication (2)⇒(3) follows immediately from the fact that every system has a minimal subsystem.

It is clear that (4) implies (1); the converse implication follows from the "compactness principle" discussed in Section 1.

To prove that (1)⇒(5), notice that any syndetic set S induces a coloring of $\mathbb{N}$ by covering it with finitely many shifts; since $\mathcal{P}$ is shift invariant, if $S - i$ contains a configuration in $\mathcal{P}$, then so does S. Conversely, if (5) holds, then in view of Lemma 8.19 so does (2).

Using Lemma 8.18 we deduce that (6)⇒(1). It is trivial that (7)⇒(6) so to finish the proof it will suffice to show that (2)⇒(7). Let $A = S \cap T$ for a syndetic S and a thick T. For each $N \in \mathbb{N}$ let $m_N \in \mathbb{N}$ such that $\{m_N, \dots, m_N + N\} \subset T$.

Consider the left shift $T : \{0, 1\}^{\mathbb{N}_0} \to \{0, 1\}^{\mathbb{N}_0}$ and let $X \subset \{0, 1\}^{\mathbb{N}_0}$ be the orbit closure of the point 1_A. Passing to a subsequence of (m_N) if needed, we can assume that the limit $y = \lim_{N\to\infty} T^{m_N} 1_A$ exists. Then $y \in X$ and hence the orbit closure X_1 of y is a subsystem of X. It can be proved that the point $(0, 0, \dots)$ does not belong to X_1 (cf. Exercises 8.24 and 8.25 below). Therefore the clopen set $U := \{x \in X : x_0 = 1\}$ has non-empty intersection with any subsystem of X_1, and in particular U has non-empty intersection with a minimal subsystem Y of (X, T).

Using part (2) on the open subset $U \cap Y$ of Y we find a configuration $C \in \mathcal{P}$ such that $W := \bigcap_{i \in C} T^{-i}U$ satisfies $W \cap Y \neq \emptyset$. We can now apply Lemma 8.21 to deduce that $B := \{n \in \mathbb{N} : T^n 1_A \in W\}$ is piecewise syndetic. For every $n \in B$ and $i \in C$ we have $T^n 1_A \in T^{-i}U$, so $T^{n+i}1_A \in U$ so $n+i \in A$. We conclude that $n + C \subset A$ and this finishes the proof. □

Exercise 8.24. *Show that the point y constructed at the end of the proof of Theorem 8.23 is the indicator function of a syndetic set.*

Exercise 8.25. *Show that if $y \in \{0,1\}^{\mathbb{N}_0}$ is the indicator function of a syndetic set then $(0, 0, \dots)$ does not belong to the orbit closure of y under the shift.*

Combining Theorem 8.23 with van der Waerden's theorem we obtain the following strengthening.

Corollary 8.26. *Let $A \subset \mathbb{N}$ be piecewise syndetic and let $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that the intersection $A \cap (A - n) \cap \cdots \cap (A - kn)$ is piecewise syndetic.*

Proof. Theorem 1.4 can be reformulated as stating that $\mathcal{P} := \{\{x, x+n, \dots, x+kn\} : x, n \in \mathbb{N}\}$ is monochromatic. In view of condition (7) in Theorem 8.23, there exists $x, n \in \mathbb{N}$ such that the set $B := \{m \in \mathbb{N} : m+\{x, x+n, \dots, x+kn\} \subset A\}$ is piecewise syndetic. The desired conclusion now follows from the observation that $A \cap (A - n) \cap \cdots \cap (A - kn) \supset B + x$. □

8.4. **Monochromatic sums and products.** In this subsection we use the facts established above to prove the following theorem.

Theorem 8.27. *If $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exists $x, y \in \mathbb{N}$ and $t \in \{1, \dots, r\}$ such that*

$$\{x, x + y, xy\} \subset C_t. \tag{8.1}$$

Proof. We will construct inductively four sequences:

- an increasing sequence $(y_i)_{i \geq 1}$ of natural numbers,
- two sequences $(B_i)_{i \geq 0}$ and $(D_i)_{i \geq 1}$ of piecewise syndetic subsets of $\mathbb{N}$,
- a sequence $(t_i)_{i \geq 0}$ of colors in $\{1, \dots, r\}$,

such that $B_i \subset C_{t_i}$ for every $i \geq 0$.

Initiate by choosing $t_0 \in \{1, \dots, r\}$ such that C_{t_0} is piecewise syndetic, and let $B_0 := C_{t_0}$. Assume now that $i \geq 1$ and that we have already defined $(t_j)_{j=0}^{i-1}$, $(y_j)_{j=1}^{i-1}$, $(B_j)_{j=0}^{i-1}$ and $(D_j)_{j=1}^{i-1}$. We apply Corollary 8.26 to find $y_i \in \mathbb{N}$ such that

$$D_i := B_{i-1} \cap \bigcap_{j=1}^{i} \Big(B_{i-1} - y_j^2 \cdots y_{i-1}^2 y_i\Big) \tag{8.2}$$

is piecewise syndetic (with the convention that for $i = j$, the (empty) product $y_j^2 \cdots y_{i-1}^2$ equals 1). Observe that $y_i D_i$ is also piecewise syndetic, and therefore Lemma 8.18 provides some $t_i \in \{1, \dots, r\}$ such that $B_i := y_i D_i \cap C_{t_i}$ is piecewise syndetic. This finishes the construction of the sequences.

Note that $B_i \subset y_i D_i \subset y_i B_{i-1}$; iterating this fact we obtain

$$\forall\, 0 \leq j < i, \qquad B_i \subset y_{j+1} y_{j+2} \cdots y_i B_j. \tag{8.3}$$

Since the sequence (t_i) takes only finitely many values, there exist (infinitely many) $j < i$ such that $t_i = t_j$. Let $\tilde{x} \in B_i$, let $y := y_{j+1} \cdots y_i$, and let $x := \tilde{x}/y$. We claim that $\{x, x + y, xy\} \subset C_{t_i}$, which will complete the proof. Indeed $xy = \tilde{x} \in B_i \subset C_{t_i}$ and from (8.3) we have $xy \in B_i \subset yB_j$ so $x \in B_j \subset C_{t_j} = C_{t_i}$. Finally we have

$$\begin{array}{rcl} y(x+y) & = & \tilde{x} + y^2 \in B_i + y^2 \subset y_i D_i + y^2 \\ \text{using (8.2)} & \subset & y_i\big(B_{i-1} - y_{j+1}^2 \cdots y_{i-1}^2 y_i\big) + y^2 \\ \text{using (8.3)} & \subset & y_i\big(y_{j+1} \cdots y_{i-1} B_j - y_{j+1}^2 \cdots y_{i-1}^2 y_i\big) + y^2 \\ & = & yB_j - y^2 + y^2 = yB_j, \end{array}$$

which implies that $x + y \in B_j \subset C_{t_j} = C_{t_i}$. □