9. Infinite Ramsey Theory

So far we have discussed only finite configurations that appear monochromatically whenever one partitions $\mathbb{N}$, or in every set with positive upper density. In this section, we mention some infinite configurations that have similar properties.

We start by recalling Ramsey's theorem. Given a set X and $m \in \mathbb{N}$, denote by $\binom{X}{m} := \{A \subset X : |A| = m\}$ the collection of all subsets of X with exactly m elements. Recall that a complete graph is a pair $(V, \binom{V}{2})$ for a set V.

Theorem 9.1 (Ramsey's theorem for graphs)**.** *For any $r, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that whenever V is a set with $|V| = n$ and we finitely color the edges of the complete graph with vertices V, i.e. $\binom{V}{2} = C_1 \cup \cdots \cup C_r$, there exists a color C_i and a set $S \subset V$ with $|S| = k$ such that $\binom{S}{2} \subset C_i$.*

There is an infinite version of Ramsey's theorem, which is the version we will be interested in. For convenience, we use $\mathbb{N}$ to stand for the countably infinite set of vertices, but naturally any other set of the same cardinality could play the same role.

Theorem 9.2 (Infinite Ramsey's theorem for graphs)**.** *Whenever $\binom{\mathbb{N}}{2} = C_1 \cup \cdots \cup C_r$, there exists a color C_i and an infinite subset $S \subset \mathbb{N}$ such that $\binom{S}{2} \subset C_i$.*

As is often the case, the infinite version has a shorter formulation, and it implies the finitistic formulation.

Exercise 9.3. *Show that Theorem 9.2 implies Theorem 9.1.*

However, unlike in many other Ramsey theoretic examples, the finite and infinite versions are not equivalent. This is made precise by the Paris-Harrington Theorem.

As a corollary of Theorem 9.2 we obtain the following arithmetic corollary. For a set $I \subset \mathbb{N}$ we denote by $I \oplus I$ the ***restricted sumset*** $\{x + y : x, y \in I; x \neq y\}$.

Corollary 9.4. *For any finite coloring of $\mathbb{N}$ there exists an infinite set $I \subset \mathbb{N}$ such that $I \oplus I$ is monochromatic.*

Proof. Suppose $\mathbb{N} = C_1 \cup \cdots \cup C_r$. Let $\tilde{C}_i := \left\{\{a, b\} \in \binom{\mathbb{N}}{2} : a + b \in C_i\right\}$. Observe that $\binom{\mathbb{N}}{2} = \tilde{C}_1 \cup \cdots \cup \tilde{C}_r$, so we can apply Theorem 9.2 to find an infinite set $I \subset \mathbb{N}$ such that $\binom{I}{2}$ is contained in a single $\tilde{C}_i$. But this means that $I \oplus I \subset C_i$. □

Exercise 9.5. *Find a 3-coloring of $\mathbb{N}$ without a monochromatic set of the form $I + I = \{x + y : x, y \in I\}$. [Hint: Color $\mathbb{N}$ with longer and longer intervals of alternating colors to avoid any pair $\{x, 2x\}$. By using the third color one can avoid pairs $\{x + y, 2x\}$ when $y \ll x$.]*

Ramsey's theorem has a version for hypergraphs:

Theorem 9.6 (Infinite Ramsey's theorem for hypergraphs)**.** *Let $m \in \mathbb{N}$. Whenever $\binom{\mathbb{N}}{m} = C_1 \cup \cdots \cup C_r$, there exists a color C_i and an infinite subset $S \subset \mathbb{N}$ such that $\binom{S}{m} \subset C_i$.*

Similarly, we can extend Corollary 9.7.

Corollary 9.7. *Let $m \in \mathbb{N}$. For any finite coloring of $\mathbb{N}$ there exists an infinite set $I \subset \mathbb{N}$ such that $I^{\oplus m} := \{x_1 + \cdots + x_m : x_1, \ldots, x_m \in I, x_1 < \cdots < x_m\}$ is monochromatic.*

I turns out that a much more spectacular result than Corollary 9.7 holds. To state we need the concept of an IP-set

Definition 9.8. *A set $A \subset \mathbb{N}$ is an* **IP-set** *if there exists an infinite set $I \subset \mathbb{N}$ such that*

$$A = \left\{ \sum_{n \in F} n : F \subset I; 0 < |F| < \infty \right\}.$$

Note that an equivalent characterization of IP-sets are sets of the form $\bigcup_{m=1}^{\infty} I^{\otimes m}$.

Theorem 9.9 (Hindman)**.** *For any finite coloring of $\mathbb{N}$ there is a monochromatic IP-set.*

Exercise 9.10. *Show that Theorem 9.9 is equivalent to the statement that any finite coloring of an IP-set yields a monochromatic IP-set. [Hint:* $\mathbb{N}$ *is the IP-set generated by the set* $I = \{2^n : n \in \mathbb{N}\}$*.]*

We will present a very simple proof of Theorem 9.9 based on the existence of idempotent ultrafilters on $\mathbb{N}$.

Definition 9.11. *An* **idempotent ultrafilter** *is a collection* p *of subsets of* $\mathbb{N}$ *satisfying*

(1) $\emptyset \notin p$ *and* $\mathbb{N} \in p$,
(2) *If* $A \in p$ *and* $B \supset A$*, then* $B \in p$,
(3) *If* $A, B \in p$ *then* $A \cap B \in p$.
(4) *If* $A \in p$ *and* $A = A_1 \cup \cdots \cup A_r$ *then one of the* A_i *is in* p.
(5) $A \in p$ *if and only if* $\{n \in \mathbb{N} : A - n \in p\} \in p$.

A collection satisfying conditions (1)-(3) is called a ***filter***. Given a set $A \subset \mathbb{N}$, the collection $p := \{B \subset \mathbb{N} : A \subset B\}$ is a filter. A collection satisfying conditions (1)-(4) is called an ***ultrafilter***. Ultrafilters are maximal filters (for the inclusion relation); therefore Zorn's lemma implies that any filter is contained in an ultrafilter. Given $n \in \mathbb{N}$, the collection $p_n := \{A \subset \mathbb{N} : n \in A\}$ is an ultrafilter. Ultrafilters of that form are called ***principal***. The existence of non-principal ultrafilters requires at least some weak form of the axiom of choice.

Proposition 9.12. *There exist idempotent ultrafilters. In fact, for any IP-set* $A \subset \mathbb{N}$ *there exists an idempotent ultrafilter* p *such that* $A \in p$.

The proof of Proposition 9.12 requires some background on ultrafilters and is beyond the scope of this lecture. We will take it for granted and use it to give a quick proof of Hindman's theorem.

Proof of Theorem 9.9. Using Proposition 9.12, let p be an idempotent ultrafilter. Given a finite coloring of $\mathbb{N}$ there is a color, call it C which belongs to p. Now let $A_1 = C$ and choose $x_1 \in A_1$ such that $A_1 - x_1 \in p$; such x_1 exists because the set $A_1 \cap \{x \in \mathbb{N} : A_1 - x \in p\}$ is in p and hence non-empty. Let $A_2 := A_1 \cap (A_1 - x_1)$ and note that $A_2 \in p$. We now proceed recursively for each $n = 2, 3, \ldots$, finding $x_n \in A_n$ such that $A_n - x_n \in p$ (which exists because $A_n \cap \{x \in \mathbb{N} : A_n - x \in p\}$) and letting $A_{n+1} = A_n \cap (A_n - x_n) \in p$.

We claim that for any nonempty $F \subset \mathbb{N}$, the sum $\sum_{i \in F} x_i \in A$ which finishes the proof. To prove the claim, let $n = \max F$ and $\tilde{F} = F \setminus \{n\}$ and note that $x_n \in A_n \subset A - \sum_{i \in \tilde{F}} x_i$, so that $\sum_{i \in F} x_i = x_n + \sum_{i \in \tilde{F}} x_i \in A$ □

Exercise 9.13. *Let* $k \in \mathbb{N}$ *and let* A *be an IP-set. Show that* A *contains a multiple of* k.

Observe that, in view of Exercise 9.13, the most obvious density analogue of Hindman's theorem is false, i.e. there are sets with positive upper density which do not contain an IP-set. However, these so-called local obstructions can be easily avoided if one is allowed to shift to the following question of Erdős:

Question 9.14. *Is is true that whenever* $A \subset \mathbb{N}$ *has positive upper density, there exists* $t \in \mathbb{N}$ *such that* $A - t$ *contains an IP-set?*

It turns out that even this density version of Hindman's theorem is false, and the answer to Question 9.14 is negative.

Exercise 9.15. *Show that for every* $\epsilon > 0$ *there exists a set* $A \subset \mathbb{N}$ *with upper density* $\bar{d}(A) > 1 - \epsilon$ *and such that for any* $t \in \mathbb{N}$ *there exists* $k = k(t)$ *such that* $A - t$ *has no multiples of* k.

In view of the negative answer to Question 9.14 (obtained by combining Exercises 9.13 and 9.15) Erdős made the following conjecture, which is still open.

Conjecture 9.16. *If* $A \subset \mathbb{N}$ *has* $\bar{d}(A) > 0$*, there exists an infinite set* $B \subset \mathbb{N}$ *and a shift* $t \in \mathbb{N}$ *such that* $A - t \supset B \oplus B$.

Exercise 9.17. *Find a set* $A \subset \mathbb{N}$ *with* $\bar{d}(A) > 0$ *and such that for any* t *and any infinite set* $B \subset \mathbb{N}$*,* $A - t$ *does not contain* $B + B$*, and hence a restricted sum* $B \oplus B$ *is required in Conjecture 9.16.*

Exercise 9.18. *Show that if Conjecture 9.16 holds, then one can always take* $t \in \{0, 1\}$.

Erdős made another related conjecture.

Conjecture 9.19. *If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then there are infinite sets $B, C \subset \mathbb{N}$ such that $A \supset B + C$.*

Exercise 9.20. *Show that Conjecture 9.16 implies Conjecture 9.19.*

Conjecture 9.19 has been recently established in [21] using ergodic theory. The full proof is beyond the scope of this notes, but we will look at one special case that was established earlier in [6].

One could try to use Furstenberg's correspondence principle in the form of Theorem 3.4. Note that

$$\exists B, C \subset \mathbb{N}: \ |B|, |C| = \infty, \ B + C \subset A \qquad \iff \qquad \exists B \subset \mathbb{N}: \ |B| = \infty, \ \left| \bigcap_{b \in B} A - b \right| = \infty.$$

Unfortunately Theorem 3.4 as stated does not allow one to take infinite intersections of the form $\bigcap_{b \in B} A - b$. This issue could actually be addressed; however the problem is that in a probability space (and hence in a measure preserving system) infinite sets of measure zero are negligible, so in order to obtain from the Correspondence Principle a conclusion of the form $\left|\bigcap_{b \in B} A - b\right| = \infty$ one would need to show that the corresponding set $\bigcap_{b \in B} T^{-b} A$ has positive measure. However, this is not always the case.

Exercise 9.21. *Consider the doubling map (i.e. the transformation $x \mapsto 2x \bmod 1$ on $[0, 1)$ with the Lebesgue measure) and let $A = [0, 1/2)$. Show that for any infinite set $B \subset \mathbb{N}$, the intersection $\bigcap T^{-b} A$ has zero measure.*

Exercise 9.22. $(*)$ *Consider the doubling map (i.e. the transformation $x \mapsto 2x \bmod 1$ on $[0, 1)$ with the Lebesgue measure) and let $A \subset [0, 1)$ be any Borel set. Show that for any infinite set $B \subset \mathbb{N}$, the intersection $\bigcap T^{-b} A$ has zero measure.*

As these exercises illustrate, we can not, in general, use the Correspondence Principle in the form presented in Theorem 3.4 (although it turns out that one can still use a different version of the more general Correspondence Principle encapsulated in the beginning of Section 3). However, as mentioned above, Theorem 3.4 suffices to prove a special case of Conjecture 9.19.

Definition 9.23. *A set $E \subset \mathbb{N}$ is* **weak-mixing** *if there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ satisfying the conclusion of Theorem 3.4 (i.e. $\mu(A) = \bar{d}(E)$ and (3.1) holds), such that the function $1_A - \mu(A)$ is a weak-mixing function (cf. Definition 5.15).*

Note that a weak-mixing function in a m.p.s. must have 0 integral, which is why we consider the function $1_A - \mu(A)$ instead of 1_A in this definition. In this sense the Jacobs-de Leeuw-Glicksberg decomposition (cf. Theorem 5.24) of 1_A is given by the sum of a weak mixing function and a constant function.

The following special case of Theorem 3.4 was first obtained in [6].

Theorem 9.24. *Let $A \subset \mathbb{N}$ with $\bar{d}(A) > 0$. If A is weak mixing, then there exist infinite sets $B, C \subset \mathbb{N}$ such that $A \supset B + C$.*

The relevance of the weak mixing property is captured by the following property.

Exercise 9.25. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $A \in \mathcal{B}$ be such that $\mu(A) > 0$ and the function $1_A - \mu(A)$ is weak mixing. Show that for any $B \in \mathcal{B}$ with $\mu(B) > 0$, the set*

$$R := \{n \in \mathbb{N} : \mu(A \cap T^{-n} B) > 0\}$$

has full natural density, i.e. $d(R) = 1$ (which is stronger than just $\bar{d}(R) = 1$).

Proof of Theorem 9.24. Let $A \subset \mathbb{N}$ have $\bar{d}(A) > 0$ and be weak mixing. Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $D \in \mathcal{B}$ be such that $\mu(D) = \bar{d}(A)$, (3.1) holds and the function $1_D - \mu(D)$ is weak mixing. Using Exercise 9.25 it follows that for any set $E \in \mathcal{B}$ with $\mu(E) > 0$ we have

$$d\big(\{n \in \mathbb{N} : \mu(T^{-n} D \cap E) > 0\}\big) = 1. \tag{9.1}$$

We will construct recursively two increasing sequences $(b_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$ we have

$$b_n \in \bigcap_{i=1}^{n-1} (A - c_i), \qquad c_n \in \bigcap_{i=1}^{n} (A - b_i), \qquad \mu\left(\bigcap_{i=1}^{n} T^{-c_i} D\right) > 0, \qquad \mu\left(\bigcap_{i=1}^{n} T^{-b_i} D\right) > 0. \tag{9.2}$$

The first two properties in (9.2) imply that the sets $B := \{b_n : n \in \mathbb{N}\}$ and $C := \{c_n : n \in \mathbb{N}\}$ satisfy $B + C \subset A$ which is the desired conclusion.

Let $b_1 \in \mathbb{N}$ be arbitrary and let $c_1 \in A - b_1$ be arbitrary. Note that (9.2) holds for $n = 1$. Next suppose that $m > 1$ and $b_1, \ldots, b_{m-1}$ and $c_1, \ldots, c_{m-1}$ have been chosen satisfying (9.2) for every $n < m$. Using Eq. (9.1) with $E = \bigcap_{i=1}^{m-1} T^{-b_i} D$ (which from (9.2) has positive measure) it follows that $R := \{b \in \mathbb{N} : \mu(T^{-b} D \cap E) > 0\}$ has full density. In view of the correspondence property (3.1), and then again the induction hypothesis (9.2),

$$\bar{d}\left(\bigcap_{i=1}^{m-1}(A - c_i)\right) \geq \mu\left(\bigcap_{i=1}^{m-1} T^{-c_i} D\right) > 0$$

Therefore the intersection $R \cap \bigcap_{i=1}^{m-1}(A - c_i)$ has positive upper density, and in particular it is infinite. Choose $b_m > b_{m-1}$ in that intersection. With this choice of b_m, both the first and last property in (9.2) hold for $n = m$. Next use Eq. (9.1) with $E = \bigcap_{i=1}^{m-1} T^{-c_i} D$ and again the correspondence principle and the induction hypothesis to conclude that the intersection

$$\bigcap_{i=1}^{m}(A - b_i) \ \cap \ \left\{c \in \mathbb{N} : \mu\left(T^{-c} D \cap \bigcap_{i=1}^{m-1} T^{-c_i} D\right) > 0\right\}$$

has positive upper density. Choosing $c_m > c_{m-1}$ in this intersection, both the second and third properties in (9.2) are satisfied for $n = m$. This finishes the construction of the sequences (b_n) and (c_n) and hence the proof. □

It turns out that the proof of Theorem 9.24 can be simplified to yield a stronger result.

Exercise 9.26. *Show that any weak-mixing set $A \subset \mathbb{N}$ with positive upper density contains $B \otimes B$ for some infinite set $B \subset \mathbb{N}$.*

At this point, it would seem natural to prove Conjecture 9.19 by using the Jacobs-de Leeuw-Glicksberg decomposition together with Theorem 9.24 and an analysis of Kronecker systems. However, a delicate subtlety related to the fact that the Jacobs-de Leeuw-Glicksberg decomposition produces measurable but not necessarily continuous components prevents this approach from working directly.

The interested reader may find the full proof in the original manuscript [21], or in a more streamlined ergodic rendition discovered by Host in [15]. Both proofs make use (in a way or another) of methods from Ergodic Ramsey Theory, including a Correspondence Principle, and the Jacobs-de Leeuw-Glicksberg decomposition.

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