Geometry of Numbers

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Simon L Rydin Myerson

University of Warwick

For link to course webpage see `https://maths.fan`

$$x \cdot y = 0$$

$$Z(N) = \#\{x, y \in \mathbb{Z}^n \setminus \{0\} : x \cdot y = 0, \|x\| \cdot \|y\| \leq N\}$$

$\sim c_n N^{n-1} \log N \quad (c_n > 0)$

$\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$

Lemma

$$Z(N) = \sum_{x \in \mathbb{Z}^n \setminus \{0\}, \|x\| \leq \sqrt{N}} \#\{y : y \cdot x = 0, \|x\| \leq \|y\| \leq N/\|x\|\}$$
$$+ \sum_{x' \in \mathbb{Z}^n \setminus \{0\}, \|x'\| \leq \sqrt{N}} \#\{y' : y' \cdot x = 0, \|x'\| < \|y'\| \leq N/\|x'\|\}$$

$\|x\| \, \|y\| \leq N$

Proof.

If $\|x\| \leq \|y\|$ deduce $\|x\| \leq \sqrt{N}$, this is the first term.
If $\|x\| > \|y\|$ let $x' = y, y' = x$. Then deduce $\|x'\| \leq \sqrt{N}$, this is the second term. □

$\|x\| = \|y\|$

$$Z(N) = \sum_{x \in \mathbb{Z}^n \setminus \{0\}, \|x\| \le \sqrt{N}} \#\{y : y \cdot x = 0, \|x\| \le \|y\| \le N/\|x\|\}$$
$$+ \#\{y : y \cdot x = 0, \|x\| < \|y\| \le N/\|x\|\}$$

Lemma

Let $\gcd(z) = 1$. *Either let* $B_n(R) = \{y \in \mathbb{R}^n : \|y\| \le R\}$ *or* $\{y \in \mathbb{R}^n : \|y\| < R\}$. *In either case* $\# L(z)^\perp \cap B(R) = \frac{\mathrm{Vol}(B_{n-1}(R))}{\|z\|} + O(1 + \frac{R}{\lambda_1(L(z)^\perp)})^{n-2}$.

Proof.

- $\det(L(z)^\perp) = \|z\|$, $\operatorname{rank} L(z)^\perp = n-1$, $\prod_{i=1}^{n-1} \lambda_i(L(z)^\perp) \asymp_n \det(L(z)^\perp)$,
- $\# L(z)^\perp \cap B(R) = \frac{\mathrm{Vol}(B_{n-1}(R))}{\det L(z)^\perp}(1 + O_n(\frac{\lambda_{n-1}(L)}{R}))$ if $R > \lambda_{n-1}(L(z)^\perp)$,
- $\# L(z)^\perp \cap B(R) \asymp_n \prod_{i=1}^{n-1}(1 + \frac{R}{\lambda_i(\# L(z)^\perp)})$.
- If $\lambda_{n-1}(L(z)^\perp) \le R$ then

□

$x_{n-1} > \varepsilon R$

$(*)$ — $\# B_n(R) \cap L(z)^{\perp}, \quad \frac{R^{n-1}}{x_1 \cdots x_{n-1}} \ll_n \left(1 + \frac{R}{x_1}\right)^{n-2}$

$$(*) \ll_n \prod_{i=1}^{n-1} \left(1 + \frac{R}{x_i}\right) \ll_n \left(1 + \frac{R}{x_1}\right)^{n-2} \left(1 + c^{-1}\right)$$

$$\frac{R^{n-1}}{x_1 \cdots x_{n-1}} \leq \frac{R^{n-1}}{x_1^{n-2} \varepsilon R} \ll \left(1 + \frac{R}{x_1}\right)^{n-2}$$

$$Z(N) = \sum_{x \in \mathbb{Z}^n \setminus \{0\}, \|x\| \le \sqrt{N}} \#\{y : y \cdot x = 0, \|x\| \le \|y\| \le N/\|x\|\} + \#\{y : y \cdot x = 0, \|x\| < \|y\| \le N/\|x\|\},$$

$\#(L(z)^{\perp} \cap B_n(N/\|x\|) \setminus B_n(\|x\|))$

$$\#L(z)^{\perp} \cap B(R) = \frac{\mathrm{Vol}(B_{n-1}(R))}{\|z\|} + O_n\left(1 + \frac{R}{\lambda_1(L(z)^{\perp})}\right)^{n-2},$$ put $g = \gcd(x)$, $z = x/g$, get

$N^{n-1} = o(N^{n-1}\log N)$

$$Z(N) = 2 \sum_{g \in \mathbb{N}} \sum_{\substack{z \in \mathbb{Z}^n \\ \|z\| \le \sqrt{N}/g \\ \gcd(z)=1}} \left(\frac{\mathrm{Vol}(B_{n-1}(N/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|} + O_n(N/g\|z\|)^{n-2} \right).$$

$\lambda_1 \ge 1$ so

$1 + \frac{N/g\|z\|}{\lambda_1} \lesssim 1 + N/\|z\| \lesssim N/g\|z\|$ as $\frac{N}{\|z\|} \ge \sqrt{N}$

Moreover

$$\sum_{g \in \mathbb{N}} \sum_{\substack{z \in \mathbb{Z}^n \\ \|z\| \le \sqrt{N}/g}} (N/g\|z\|)^{n-2} \ll_n \sum_{g \in \mathbb{N}} (N/g)^{n-2} (\sqrt{N}/g)^2 \ll_n N^{n-1} \sum_{g \in \mathbb{N}} g^{-n} \ll N^{n-1}.$$

$1 + \frac{g\|z\|}{\lambda_1} \le 1 + \frac{N/g\|z\|}{\lambda_1}$ as $g\|z\| \le \frac{N}{g\|z\|}$

$$Z(N) = 2\sum_{g\in\mathbb{N}} \sum_{\substack{z\in\mathbb{Z}^n \\ \|z\|\le\sqrt{N}/g \\ \gcd(z)=1}} \left(\frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|} + O_n(N/g\|z\|)^{n-2} \right),$$

$$\sum_{g\in\mathbb{N}} \sum_{\substack{z\in\mathbb{Z}^n \\ \|z\|\le\sqrt{N}/g}} (N/g\|z\|)^{n-2} \ll_n N^{n-1},$$

so

$$Z(N) = O_n(N^{n-1}) + 2\sum_{g\in\mathbb{N}} \sum_{\substack{z\in\mathbb{Z}^n \\ \|z\|\le\sqrt{N}/g \\ \gcd(z)=1}} \frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|}.$$

$$Z(N) = O_n(N^{n-1}) + 2 \sum_{g \in \mathbb{N}} \sum_{\substack{z \in \mathbb{Z}^n \\ \|z\| \leq \sqrt{N}/g \\ \gcd(z)=1}} \frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|}.$$

Now we use the fact that for $n \geq 3$,

$$\#\{z \in \mathbb{Z}^n : \gcd(z) = 1, \|z\| \leq R\} = \frac{\mathrm{Vol}(B_n(R))}{\zeta(n)} + O(R^{n-1}).$$

To prove this one uses $\sum_{d \in \mathbb{N}, d|z} \mu(d) = \begin{cases} 1 & \gcd(z) = 1 \\ 0 & \gcd(z) > 1 \end{cases}$, $\quad \sum_{d \in \mathbb{N}} \mu(d) d^{-n} = \frac{1}{\zeta(n)}$.

$$\sum_{\substack{z \\ \|z\| \leq R}} \sum_{d|z} \mu(d) = \sum_{d \in \mathbb{N}} \mu(d) \sum_{\substack{z = dw \\ \|z\| \leq R \\ \|w\| \leq R/d}} 1 = \sum_{d \in \mathbb{N}} \# \mathbb{Z}^n \cap B\left(\frac{R}{d}\right)$$

$$Z(N) = O_n(N^{n-1}) + 2\sum_{g\in\mathbb{N}} \sum_{\substack{z\in\mathbb{Z}^n \\ \|z\|\le\sqrt{N}/g \\ \gcd(z)=1}} \frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|}.$$

Subs. $k=\|z\|^2$

Now

$$\sum_{\substack{\|z\|\le\sqrt{N}/g \\ \gcd(z)=1}} \frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|} = \sum_{k\le R^2} f(k)\#\{z : \gcd(z)=1, \|z\|^2 = k\},$$

$k\in\mathbb{N}$

where

$$f(k) = \frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/gk^{1/2})) - \mathrm{Vol}(B_{n-1}(gk^{1/2}))}{k^{1/2}} = \frac{\mathrm{Vol}(B_{n-1}(1))}{k^{1/2}}\left(\left(\frac{N}{g^2k}\right)^{\frac{n-1}{2}} - gk^{\frac{n-1}{2}}\right).$$

We can use integration by parts to bring in

$\sum_{k\le R^2} \#\{\gcd(z)=1, \|z\|^2=k\}$

$$\#\{z\in\mathbb{Z}^n : \gcd(z)=1, \|z\|\le R\} = \frac{\mathrm{Vol}(B_n(R))}{\zeta(n)} + O(R^{n-1}).$$

$$Z(N) = O_n(N^{n-1}) + 2 \sum_{g \in \mathbb{N}} \sum_{\substack{z \in \mathbb{Z}^n \\ \|z\| \le \sqrt{N}/g \\ \gcd(z)=1}} \frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|}.$$

Using what Wikipedia calls Abel summation (a form of integration by parts),

$$\sum_{k \le R^2} f(k) \#\{z : \gcd(z) = 1, \|z\|^2 = k\} =$$

$$f(R^2)\#\{z : \gcd(z) = 1, \|z\|^2 \le R^2\} - \int_1^{R^2} f'(u)\#\{z : \gcd(z) = 1, \|z\|^2 \le u\}\, du =$$

$$f(R^2)\frac{\mathrm{Vol}(B_n(R))}{\zeta(n)} - \int_1^{R^2} f'(u)\frac{\mathrm{Vol}(B_n(u^{1/2}))}{\zeta(n)}\, du + O\left(|f(R^2)|R^{n-1} + \int_1^{R^2} |f'(u)|u^{(n-1)/2}\, du\right).$$

Here we use the fact that for $n \ge 3$,

$$\#\{z \in \mathbb{Z}^n : \gcd(z) = 1, \|z\| \le R\} = \frac{\mathrm{Vol}(B_n(R))}{\zeta(n)} + O(R^{n-1}).$$

$$Z(N) = O_n(N^{n-1}) + 2\sum_{g\in\mathbb{N}} \sum_{\substack{z\in\mathbb{Z}^n \\ \|z\|\le\sqrt{N}/g \\ \gcd(z)=1}} \frac{\mathrm{Vol}(B_{n-1}(\sqrt{N}/g\|z\|)) - \mathrm{Vol}(B_{n-1}(g\|z\|))}{\|z\|}.$$

Using integration by parts,

$$\sum_{k\le R^2} f(k)\#\{z : \gcd(z) = 1, \|z\|^2 = k\} =$$

$$f(R^2)\frac{\mathrm{Vol}(B_n(R))}{\zeta(n)} - \int_1^{R^2} f'(u)\frac{\mathrm{Vol}(B_n(u^{1/2}))}{\zeta(n)}\,du + O\left(|f(R^2)|R^{n-1} + \int_1^{R^2} |f'(u)|u^{(n-1)/2}\,du\right)$$

$$= \frac{\mathrm{Vol}(B_n(1))}{\zeta(n)} \int_1^{R^2} f(u)\frac{n}{2}u^{\frac{n}{2}-1}\,du + O\left(|f(1)| + |f(R^2)|R^{n-1} + \int_1^{R^2} |f'(u)|u^{(n-1)/2}\,du\right).$$

$$Z(N) = O_n(N^{n-1}) + 2\sum_{g\in\mathbb{N}}\sum_{k\le R^2} f(k)\#\{z : \gcd(z) = 1, \|z\|^2 = k\},$$

where

$$f(k) = \frac{\mathrm{Vol}(B_{n-1}(1))}{k^{1/2}}\left(\left(\frac{N}{g^2k}\right)^{\frac{n-1}{2}} - gk^{\frac{n-1}{2}}\right).$$

$$\sum_{k\le R^2} f(k)\#\{z : \gcd(z) = 1, \|z\|^2 = k\} =$$

$$\frac{\mathrm{Vol}(B_n(1))}{\zeta(n)}\int_1^{R^2} f(u)\frac{n}{2}u^{\frac{n}{2}-1}\,du + O\left(|f(1)| + |f(R^2)|R^{n-1} + \int_1^{R^2} |f'(u)|u^{(n-1)/2}\,du\right).$$

$$\sum_{g\in\mathbb{N}} |f(1)| + |f(R^2)|R^{n-1} + \int_1^{R^2} |f'(u)|u^{(n-1)/2}\, du$$

where

$$f(k) = \frac{\mathrm{Vol}(B_{n-1}(1))}{k^{1/2}} \left(\left(\frac{N}{g^2 k}\right)^{\frac{n-1}{2}} - gk^{\frac{n-1}{2}} \right).$$

$$2\sum_{g\in\mathbb{N}} \frac{\mathrm{Vol}(B_n(1))}{\zeta(n)} \int_1^{R^2} f(u)\frac{n}{2}u^{\frac{n}{2}-1}\,du$$

where

$$f(k) = \frac{\mathrm{Vol}(B_{n-1}(1))}{k^{1/2}} \left(\left(\frac{N}{g^2 k} \right)^{\frac{n-1}{2}} - gk^{\frac{n-1}{2}} \right).$$