Assignment 6

Due Monday 12 November 15:00 (return to your supervisor's pigeon hole)

Problem 1. Find $\sup A$ and $\inf A$ where A is the set defined by
(a) $A = \{x \in \mathbb{R} : x^4 < 16\}$
(b) $A = \{x \in \mathbb{R} : x^4 \le 16\}$
(c) $A = \{x \in \mathbb{Q} : x = \frac{1}{n} + 2^{-n}, n \in \mathbb{N}\}$
(d) $A = \{x \in \mathbb{R} : |x| < 3 \text{ and } x^2 > 2\}$
(e) $A = \{x = \frac{m}{2^n+1},\ \ m, n \in \mathbb{N}\}$
(f) $A = \{x \in \mathbb{Q} : 0 < \sqrt{x} < 3\}$

Problem 2. Prove the following theorem, originally due to Cauchy. Suppose that $(a_n) \to a$. Then the sequence (b_n) defined by

$$b_n = \frac{a_1 + a_2 + ... + a_n}{n}$$

is convergent and $(b_n) \to a$.

Problem 3. Find the limit of the sequence (a_n) defined by

$$a_n = \frac{1 + \sqrt[2]{2} + \sqrt[3]{3} + ... + \sqrt[n]{n}}{n}.$$

(The result of the previous exercise may help!)

Problem 4. Let (a_n) be a decreasing sequence that is bounded below. Does it necessarily converge? If yes, prove it. If not, give a counter-example.

Problem 5. Consider the sequence defined by

$$a_{n+1} = \frac{c}{a_n} + \frac{a_n}{2}, \qquad a_0 = 2c.$$

Here, $c > 0$ is a fixed parameter, $\frac{1}{2} < c < \frac{9}{2}$. Show the following:
(i) (a_n) is bounded below by $\sqrt{2c}$;
(ii) (a_n) is decreasing;
(iii) (a_n) is convergent (and find the limit).
(iv) Let $e_n = \frac{a_n - \sqrt{2c}}{2\sqrt{2c}}$. Notice that e_n measures the relative distance between a_n and its limit. Estimate the speed of convergence of (a_n) by proving that

$$|e_n| \le \left(\frac{\sqrt{2c} - 1}{2}\right)^{2^n}$$

Hint. For part (iv), first prove that

$$e_{n+1} \le e_n^2,$$

then find an upper bound on e_{n+1} in terms of e_1 by iterating the above inequality.

Problem 6. Give an example of a sequence (a_n) which is not convergent, but such that $a_{n+1} - a_n \to 0$.