Analysis — MA131

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Contents

Chapter 1

Inequalities

1.1 What are Inequalities?

An inequality is a statement involving one of the order relationships $<$, $>$, $\leq$, $\geq$. Inequalities can be split into two types:

(i) those whose truth depends on the value of the variables involved, e.g. $x^2 > 4$ is true if and only if $x < -2$ or $x > 2$;

(ii) those which are always true, e.g. $(x-3)^2 + y^2 \geq 0$ is true for all real values of x and y.

We begin here by looking at how to deal with inequalities of the first type. Later on in sections 1.6 and 1.7 we will see some examples of the second type.

With the first type of inequality our task is usually to find the set of values for which the inequality is true; this is called *solving* the inequality. The set we find is called the *solution set* and the numbers in the set are called *solutions*.

The basic statement $x < y$ can be interpreted in two simple ways.

$$x < y \iff y - x \text{ is positive} \iff \begin{cases} \text{The point representing } x \text{ on the standard number line is to the left of the} \\ \text{point representing } y. \end{cases}$$

> **Caution**
> 0 is not a positive number

It can be shown that with this interpretation we have the following familiar basic rules for manipulating inequalities based on "$<$". Similar definitions and rules apply to $>$, $\leq$, and $\geq$.

> **Remember these rules**
> These rules are important. You should know them by heart.

Rule	Example (based on "$<$")
Adding the same number to each side preserves the inequality.	$x < y \iff x + a < y + a$
Multiplying both sides by a positive number preserves the inequality.	If a is positive then $x < y \implies ax < ay$
Multiplying both sides by a negative number reverses the inequality.	If b is negative then $x < y \implies bx > by$
Inequalities of the same type are *transitive*.	$x < y$ and $y < z \implies x < z$.

We *solve* an inequality involving variables by finding all the values of those variables that make the inequality true. Some solutions are difficult to find and not all inequalities have solutions.

Exercise 1

1. Show that $x = 0$ is a solution of $\frac{(x-2)(x-4)}{(x+3)(x-7)} < 0$.
2. Solve the inequalities:
   (a) $x^2 > 4$; (b) $x - 2 \leq 1 + x$; (c) $-2 < 3 - 2x < 2$.
3. Write down an inequality that has no solution.

1.2 Using Graphs

Graphs can often indicate the solutions to an inequality. The use of graphs should be your first method for investigating an inequality.

Exercise 2 Draw graphs to illustrate the solutions of the following inequalities.

1. $x^3 < x$; 2. $1/x < x < 1$.

In the second case you will need to plot the graphs of $y = 1/x$, $y = x$ and $y = 1$.

1.3 Case Analysis

You solve inequalities by using the basic rules given in section 1.1. When solving inequalities which involve products, quotients and modulus signs (more on these later) you often have to consider separate cases. Have a good look at the following examples.

Examples

1. Solve $x^2 < 1$.

   First we notice that $x^2 < 1 \iff x^2 - 1 < 0 \iff (x+1)(x-1) < 0$. We can see at once that there are two possible cases:

   (a) $x + 1 > 0$ and $x - 1 < 0 \iff x > -1$ and $x < 1 \iff -1 < x < 1$;
   (b) $x + 1 < 0$ and $x - 1 > 0 \iff x < -1$ and $x > 1$ Impossible!

   It follows that $-1 < x < 1$.

2. Solve $\frac{1}{x} + \frac{1}{x+1} > 0$.

   To get an idea of the solutions of this inequality it is a good idea to draw graphs of $\frac{1}{x}$ and $\frac{-1}{x+1}$ on the same axis because $\frac{1}{x} + \frac{1}{x+1} > 0 \iff \frac{1}{x} > \frac{-1}{x+1}$. It is useful to note that $\frac{1}{x} + \frac{1}{x+1} = \frac{2x+1}{x(x+1)}$. We look for the points where the denominator changes sign (at $x = -1$ and $x = 0$) and choose our cases accordingly. For the values $x = 0$ or $x = -1$ the inequality is meaningless so we rule these values out straight away.

Products

The product xy of two real numbers is positive if and only if x and y are either *both* positive or *both* negative. Their product is negative if and only if they have opposite signs.

Sign Language

We use the double implication sign ($\iff$) to ensure that we find only the solution set and not some larger set to which it belongs. For instance, suppose we wished to solve $2x < -1$. We could quite correctly write

$$2x < -1 \implies 2x < 0 \implies x < 0,$$

but clearly it is not true that $x < 0 \implies 2x < -1$.

(a) Consider only $x < -1$. In this case x and $x+1$ are negative and $x(x+1)$ is positive. So $\frac{2x+1}{x(x+1)} > 0 \iff 2x+1 > 0 \iff x > -1/2$ which is impossible for this case.

(b) Consider only $-1 < x < 0$. Then $x(x+1)$ is negative so $\frac{2x+1}{x(x+1)} > 0 \iff 2x+1 < 0 \iff x < -1/2$. So we have solutions for the x under consideration exactly when $-1 < x < -1/2$.

(c) Consider only $x > 0$. Then $x(x+1)$ is positive so as in case 1 we require $x > -1/2$. So the solutions for those x under consideration are exactly $x > 0$.

It follows that the solution set is exactly those x such that either$-1 < x < -1/2$ or $x > 0$.

1.4 Taking Powers

Exercise 3 Is the following statement true for all x and y: "If $x < y$ then $x^2 < y^2$"? What about this statement: "If $x^2 < y^2$ then $x < y$"?

You probably suspect that the following is true:

Power Rule
If x and y are *positive* real numbers then, for each natural number n, $x < y$ if and only if $x^n < y^n$.

Example This is another way of saying that if x is positive then the function x^n is strictly increasing. We would like to prove this useful result. Of course we are looking for an arithmetic proof that does not involve plotting graphs of functions but uses only the usual rules of arithmetic. The proof must show both that $x < y \implies x^n < y^n$ *and* that $x^n < y^n \implies x < y$. Notice that these are two *different* statements.

Caution
The Power Rule doesn't work if x or y are negative.

Exercise 4

1. Use mathematical induction to prove that if both x and y are positive then $x < y \implies x^n < y^n$.

2. Now try to prove the *converse*, that if both x and y are positive then $x^n < y^n \implies x < y$.

   If inspiration doesn't strike today, stay tuned during *Foundations*, especially when the word *contrapositive* is mentioned.

Contrapositive
The contrapositive of the statement $p \implies q$ is the statement *not* $q \implies$ *not* p. These are equivalent, but sometimes one is easier to prove than the other.

1.5 Absolute Value (Modulus)

At school you were no doubt on good terms with the modulus, or absolute value, sign and were able to write useful things like $|2| = 2$ and $|-2| = 2$. What you may not have seen written explicitly is the definition of the absolute value, also known as the absolute value function.

Definition

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Exercise 5

1. Check that this definition works by substituting in a few positive and negative numbers, not to mention zero.

2. Plot a graph of the absolute value function.

Proposition
The following are key properties of the absolute value function $|\cdot|$.

1. $||x|| = |x|$.

2. $|xy| = |x||y|$.

3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

Proof. It is easy to prove 1. directly from the definition since $|x|$ in the left hand side is always positive. To prove 2. we consider the three cases, where one of x or y is negative and the other is positive, where both are positive, and when both are negative. For example, if both x and y are negative then xy is positive and

$$|xy| = xy = -|x|(-|y|) = |x||y|.$$

The proof of 3. uses the fact that

$$1 = x \cdot \frac{1}{x} \underset{\text{by 2.}}{\Longrightarrow} 1 = |x| \left|\frac{1}{x}\right| \implies \frac{1}{|x|} = \left|\frac{1}{x}\right|.$$

Now we can write

$$\left|\frac{x}{y}\right| = |x| \left|\frac{1}{y}\right| = |x| \frac{1}{|y|} = \frac{|x|}{|y|}.$$

■

In Analysis, intervals of the real line are often specified using absolute values. The following result makes this possible:

Theorem *Interval Property*
If x and b are real numbers and $b > 0$, then $|x| < b$ if and only if $-b < x < b$.

Proof. Suppose $|x| < b$. For $x \geq 0$ this means that $x < b$ and for $x < 0$ this means that $-x < b$ which is the same as $x > -b$. Together these prove half the result. Now suppose $-b < x < b$. Then $-b < |x| < b$ if $x \geq 0$ by definition. If $x < 0$ then $-b < -|x| < b$ and it follows again that $-b < |x| < b$. ■

Corollary
If y, a and b are real numbers and $b > 0$, then $|y - a| < b$ if and only if $a - b < y < a + b$.

Proof. Substitute $x = y - a$ in the interval property. ■

This corollary justifies the graphical way of thinking of the modulus sign $|a - b|$ as the distance along the real line between a and b. This can make solving simple inequalities involving the absolute value sign very easy. To solve the inequality $|x - 3| < 1$ you need to find all the values of x that are within distance 1 from the number 3, i.e. the solution set is $2 < x < 4$.

Exercise 6 Solve the inequalities:

1. $|x - 2| > 1$; 2. $|x + 5| < 3$; 3. $|6x - 12| > 3$.

[Hint: don't forget that $|x + 5| = |x - (-5)|$ is just the distance between x and -5 and that $|6x - 12| = 6|x - 2|$ is just six times the distance between x and 2.]

If the above graphical methods fail then expressions involving absolute values can be hard to deal with. Two arithmetic methods are to try to get rid of the modulus signs by Case Analysis or by squaring. We illustrate these methods in the following very simple example.

Squaring
The method of squaring depends upon the equivalence: if $b > a \geq 0$, then

$$a < |x| < b$$
$$\iff a^2 < x^2 < b^2$$

Example Solve the inequality $|x + 4| < 2$. Squaring:

$$\begin{aligned} 0 \leq |x + 4| < 2 &\iff (x + 4)^2 < 4 \\ &\iff x^2 + 8x + 16 < 4 \\ &\iff x^2 + 8x + 12 < 0 \\ &\iff (x + 2)(x + 6) < 0 \\ &\iff -6 < x < -2. \end{aligned}$$

Case Analysis:

1. Consider $x > -4$. Then $|x + 4| < 2 \iff x + 4 < 2 \iff x < -2$. So solutions for this case are $-4 < x < -2$.

2. Consider $x \leq -4$. Then $|x + 4| < 2 \iff -x - 4 < 2 \iff x > -6$. So solutions for this case are $-6 < x \leq -4$.

So the solution set is $-6 < x < -2$.

Exercise 7 Solve the following inequalities:

$$1.\ |x-1| + |x-2| \geq 5; \qquad 2.\ |x-1| \cdot |x+1| > 0.$$

1.6 The Triangle Inequality

Here is an essential inequality to put in your mathematical toolkit. It comes in handy in all sorts of places.

Dummy Variables
The Triangle Inequality holds for all values, so you can stick into it any numbers or variables you like. The x and y are just dummies.

Have A Go
You can also prove the Triangle Inequality by Case Analysis. The proof is longer, but it is a good test of whether you can really handle inequalities.

The Triangle Inequality
For all real numbers x and y, $|x+y| \leq |x| + |y|$.

Exercise 8

1. Put a variety of numbers into the Triangle Inequality and convince yourself that it really works.
2. Write out the triangle inequality when you take $x = a - b$ and $y = b - c$.
3. Prove the Triangle Inequality.
   [Hint: Square both sides.]

1.7 Arithmetic and Geometric Means

Given two numbers a and b the arithmetic mean is just the average value that you are used to, namely $(a+b)/2$. Another useful average of two positive values is given by the geometric mean. This is defined to take the value $\sqrt{ab}$.

Exercise 9 Calculate the arithmetic and the geometric mean for the numbers 0, 10 and 1, 9 and 4, 6 and 5, 5.

One motivation for the geometric mean is average growth (or average interest): Suppose that a sum S_0 is deposited on a bank account and that it receives interest r_1 at the end of the 1st year, r_2 at the end of the second year, etc... The total amount after n years is

$$S_n = S_0(1+r_1)...(1+r_n).$$

What is the average interest? The arithmetic mean $\frac{r_1+...+r_n}{n}$ is not relevant here. Rather, we are looking for the number $\bar{r}$ such that

$$S_n = S_0(1+\bar{r})^n.$$

It follows that

$$(1+\bar{r})^n = (1+r_1)...(1+r_n) \qquad \Longleftrightarrow \qquad 1+\bar{r} = \sqrt[n]{(1+r_1)...(1+r_n)}.$$

We see that $1+\bar{r}$ is given by the geometric mean of $1+r_1, \ldots, 1+r_n$.

Exercise 10

1. Show, for positive a and b, that $\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2}$.

2. Show that the arithmetic mean is always greater than or equal to the geometric mean. When can they be equal?

Definition
Suppose we have a list of n positive numbers $a_1, a_2, ..., a_n$. We can define the arithmetic and geometric means by:

Arithmetic Mean $= \frac{a_1+a_2+...+a_n}{n}$;

Geometric Mean $= \sqrt[n]{a_1 a_2 ... a_n}$.

Exercise 11

1. Calculate both means for the numbers 1, 2, 3 and for 2, 4, 8.

2. It is a true inequality that the arithmetic mean is always greater than or equal to the geometric mean. There are many proofs, none of them are straightforward. Puzzle for a while to see if you can prove this result.

   [Hint: The case $n = 4$ is a good place to start.]

The geometric mean of a and b can be computed using "arithmetic-harmonic means": Let $a_0 = a, b_0 = b$, and then

$$a_1 = \frac{a_0+b_0}{2}, \qquad b_1 = \frac{2}{\frac{1}{a_0}+\frac{1}{b_0}},$$

and

$$a_2 = \frac{a_1+b_1}{2}, \qquad b_2 = \frac{2}{\frac{1}{a_1}+\frac{1}{b_1}},$$

and so on... One can prove that the sequences (a_n) and (b_n) both converge to $\sqrt{ab}$. You are encouraged to prove this later, after Chapter 6.

1.8 * Archimedes and π *

Archimedes used the following method for calculating π.

The area of a circle of radius 1 is π. Archimedes calculated the areas

$$\begin{aligned} A_n &= \text{ area of the circumscribed regular polygon with } n \text{ sides,} \\ a_n &= \text{ area of the inscribed regular polygon with } n \text{ sides.} \end{aligned}$$

The area of the circle is between that of the inscribed and circumscribed polygons, so $a_n \leq \pi \leq A_n$ for any n. Archimedes claimed that the following two formulae hold:

$$a_{2n} = \sqrt{a_n A_n}; \qquad A_{2n} = \frac{2A_n a_{2n}}{A_n + a_{2n}}.$$

Exercise 12 What are the values of A_4 and a_4? Use Archimedes formulae and a calculator to find $a_8, A_8, a_{16}, A_{16}, a_{32}, A_{32}, a_{64}, A_{64}$. How many digits of π can you be sure of?

To prove the formulae, Archimedes used geometry, but we can find a short proof using trigonometry.

Exercise 13 Use trigonometry to show that $A_n = n \tan\left(\frac{\pi}{n}\right)$ and $a_n = \frac{n}{2} \sin\left(\frac{2\pi}{n}\right)$. Now use the double angle formulae to prove Archimedes' formulae.

Check Your Progress
By the end of this chapter you should be able to:

- Solve inequalities using case analysis and graphs.
- Define the absolute value function and manipulate expressions containing absolute values.
- Interpret the set $\{x : |x - a| < b\}$ as an interval on the real line.
- Prove that if x and y are positive real numbers and n is a natural number then $x \leq y$ iff $x^n \leq y^n$.
- State and prove the Triangle Inequality.