# Chapter 10

# Series IV

## 10.1 Rearrangements of Series

If you take any *finite* set of numbers and rearrange their order, their sum remains the same. But the truly weird and mind-bending fact about *infinite* sums is that, in some cases, you can rearrange the terms to get a totally different sum. We look at one example in detail.

The sequence

$$(b_n) = 1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{8}, \frac{1}{5}, -\frac{1}{10}, -\frac{1}{12}, \frac{1}{7}, -\frac{1}{14}, -\frac{1}{16}, \frac{1}{9}, \ldots$$

contains all the numbers in the sequence

$$(a_n) = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{9}, -\frac{1}{10}, \frac{1}{11}, -\frac{1}{12}, \frac{1}{13}, -\frac{1}{14}, \ldots$$

but rearranged in a different order: each of the positive terms is followed by not one but *two* of the negative terms. You can also see that each number in $(b_n)$ is contained in $(a_n)$. So this rearrangement effectively shuffles, or permutes, the indices of the original sequence. This leads to the following definition.

**Definition**
The sequence $(b_n)$ is a *rearrangement* of $(a_n)$ if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ (i.e. a permutation on $\mathbb{N}$) such that $b_n = a_{\sigma(n)}$ for all $n$.

**Shuffling the (Infinite) Pack**

The permutation $\sigma$ simply shuffles about the terms of the old sequence $(a_n)$ to give the new sequence $(a_{\sigma(n)})$.

**Exercise 1** What permutation $\sigma$ has been applied to the indices of the sequence $(a_n)$ to produce $(b_n)$ in the example above? Answer this question by writing down an explicit formula for $\sigma(3n), \sigma(3n-1), \sigma(3n-2)$.

**Reciprocal Rearrangements**

If $(b_n)$ is a rearrangement of $(a_n)$ then $(a_n)$ must be a rearrangement of $(b_n)$. Specifically, if $b_n = a_{\sigma(n)}$ then $a_n = b_{\sigma^{-1}(n)}$.

Don't get hung up on this exercise if you're finding it tricky, because the really interesting part comes next.

We have defined the rearrangement of a sequence. Using this definition, we say that the *series* $\sum b_n$ is a rearrangement of the *series* $\sum a_n$ if the sequence $(b_n)$ is a rearrangement of the sequence $(a_n)$.

We know already that

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \log(2)$$

We now show that our rearrangement of this series has a different sum.

**Exercise 2** Show that:

$$\sum b_n = 1 + -\frac{1}{2} + -\frac{1}{4} + \frac{1}{3} + -\frac{1}{6} + -\frac{1}{8} + \frac{1}{5} + -\frac{1}{10} + \\ -\frac{1}{12} + \frac{1}{7} + -\frac{1}{14} + -\frac{1}{16} + \frac{1}{9} + \cdots = \frac{\log 2}{2}$$

Hint: Let $s_n = \sum_{k=1}^n a_k$ and let $t_n = \sum_{k=1}^n b_k$. Show that $t_{3n} = \frac{s_{2n}}{2}$ by using the following grouping of the series $\sum b_n$:

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \ldots$$

This example is rather scary. However, for series with all positive terms it does not matter in what order you add the terms.

**Theorem**
Suppose that $a_n \geq 0$ for all $n$. Then, if $(b_n)$ is a rearrangement of $(a_n)$, we have $\sum b_n = \sum a_n$.

Remark: The theorem holds both if $\sum a_n$ is finite *or* infinite (in which case $\sum b_n = \infty$).

**Proof.** Let $s_n = \sum_{k=1}^n a_k$ and $t_n = \sum_{k=1}^n b_k$. Let $s = \lim s_n$ and $t = \lim t_n$ (either these limits exist, or $s$ or $t$ are equal to infinity). For all $n$, we have

$$s_n \leq t, \qquad t_n \leq s.$$

Indeed, $t$ involves the sum over all $b_k$, so it involves the sum over $a_1, \ldots, a_n$. Same for $s$. Taking the limit $n \to \infty$, we find that $s \leq t$ and $t \leq s$. Then $s = t$. ■

Nor does it matter what order you add the terms of an absolutely convergent series.

**Theorem**
Suppose that $\sum a_n$ is an absolutely convergent series. If $(b_n)$ is a rearrangement of $(a_n)$ then $\sum b_n$ is absolutely convergent and $\sum b_n = \sum a_n$.

**Proof.** We know from the previous theorem that $\sum |b_k| = \sum |a_k| < \infty$. Consider

$$\sum_{k=1}^n b_k = \sum_{k=1}^n |b_k| - \sum_{k=1}^n (|b_k| - b_k).$$

Both sums of the right side involve nonnegative terms, and they are absolutely convergent. Then we can rearrange their terms, so that they converge to the limit of

$$\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} (|a_k| - a_k) = \sum_{k=1}^{n} a_k.$$

This proves that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k$. ■

In 1837 the mathematician Dirichlet discovered which type of series could be rearranged to give a different sum and the result was displayed in a startling form in 1854 by Riemann. To describe their results we have one final definition.

**Definition**
The series $\sum a_n$ is said to be *conditionally convergent* if $\sum a_n$ is convergent but $\sum |a_n|$ is not.

**Example** Back to our familiar example: $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent, because $\sum \frac{(-1)^{n+1}}{n}$ is convergent, but $\sum |\frac{(-1)^{n+1}}{n}| = \sum \frac{1}{n}$ is not.

**Exercise 3** Check from the definitions that every convergent series is either absolutely convergent or is conditionally convergent.

**Exercise 4** State with reasons which of the following series are conditionally convergent.

$$1. \sum \frac{(-1)^{n+1}}{n^2} \qquad 2. \sum \frac{\cos(n\pi)}{n} \qquad 3. \sum \frac{(-1)^{n+1} n}{1+n^2}$$

Conditionally convergent series are the hardest to deal with and can behave very strangely. The key to understanding them is the following lemma.

**Lemma**
If a series is conditionally convergent, then the series formed from just its positive terms diverges to infinity, and the series formed from just its negative terms diverges to minus infinity.

**Exercise 5** Prove this Lemma using the following steps.

1. Suppose $\sum a_n$ is conditionally convergent. What can you say about the sign of the sequences

   $$u_n = \tfrac{1}{2}(|a_n| + a_n) \quad \text{and} \quad v_n = \tfrac{1}{2}(|a_n| - a_n)$$

   in relation to the original sequence $a_n$.

2. Show that $a_n = u_n - v_n$ and $|a_n| = u_n + v_n$. We will prove by contradiction that neither $\sum u_n$ nor $\sum v_n$ converges.

3. Suppose that that $\sum u_n$ is convergent and show that $\sum |a_n|$ is convergent. Why is this a contradiction?

4. Suppose that $\sum v_n$ is convergent and use a similar argument to above to derive a contradiction.

5. You have shown that $\sum u_n$ and $\sum v_n$ diverge. Prove that they tend to $+\infty$. Use your answer to part 1. to finish the proof.

**Theorem** *Riemann's Rearrangement Theorem*
Suppose $\sum a_n$ is a conditionally convergent series. Then for every real number $s$ there is a rearrangement $(b_n)$ of $(a_n)$ such that $\sum b_n = s$.

The last lemma allows us to construct a proof of the theorem along the following lines: We sum enough positive values to get us just above $s$. Then we add enough negative values to take us back down just below $s$. Then we add enough positive terms to get back just above $s$ again, and then enough negative terms to get back down just below $s$. We repeat this indefinitely, in the process producing a rearrangement of $\sum a_n$ which converges to $s$.

**The Infinite Case**

We can also rearrange any conditionally convergent series to produce a series that tends to infinity or minus infinity.

*How would you modify the proof to show this?*

**All Wrapped Up**

Each convergent series is either conditionally convergent or absolutely convergent. Given the definition of these terms, there are no other possibilities.
This theorem makes it clear that conditionally convergent series are the *only* convergent series whose sum can be perturbed by rearrangement.

**Proof.** Let $(p_n)$ be the subsequence of $(a_n)$ containing all its positive terms, and let $(q_n)$ be the subsequence of negative terms. First suppose that $s \geq 0$. Since $\sum p_n$ tends to infinity, there exists $N$ such that $\sum_{i=1}^{N} p_i > s$. Let $N_1$ be the *smallest* such $N$ and let $S_1 = \sum_{i=1}^{N_1} p_i$. Then $S_1 = \sum_{i=1}^{N_1} p_i > s$ and $\sum_{i=1}^{N_1-1} p_i \leq s$. Thus $S_1 = \sum_{i=1}^{N_1-1} p_i + p_{N_1} \leq s + p_{N_1}$, therefore $0 \leq S_1 - s \leq p_{N_1}$.

To the sum $S_1$ we now add just enough negative terms to obtain a new sum $T_1$ which is less than $s$. In other words, we choose the smallest integer $M_1$ for which $T_1 = S_1 + \sum_{i=1}^{M_1} q_i < s$. This time we find that $0 \leq s - T_1 \leq -q_{M_1}$.

We continue this process indefinitely, obtaining sums alternately smaller and larger than $s$, each time choosing the smallest $N_i$ or $M_i$ possible. The sequence:

$$p_1, \ldots, p_{N_1}, q_1, \ldots q_{M_1}, p_{N_1+1}, \ldots p_{N_2}, q_{M_1+1}, \ldots, q_{M_2}, \ldots$$

is a rearrangement of $(a_n)$. Its partial sums increase to $S_1$, then decrease to $T_1$, then increase to $S_2$, then decrease to $T_2$, and so on.

To complete the proof we note that for all $i$, $|S_i - s| \leq p_{N_i}$ and $|T_i - s| \leq -q_{M_i}$. Since $\sum a_n$ is convergent, we know that $(a_n)$ is null. It follows that subsequences $(p_{N_i})$ and $(q_{M_i})$ also tend to zero. This in turn ensures that the partial sums of the rearrangement converge to $s$, as required.

In the case $s < 0$ the proof looks almost identical, except we start off by summing enough negative terms to get us just below $l$. ■

**Exercise 6** Draw a diagram which illustrates this proof. Make sure you include the limit $s$ and some points $S_1, T_1, S_2, T_2, \ldots$.

**Check Your Progress**
By the end of this chapter you should be able to:

- Define what is meant by the *rearrangement* of a sequence or a series.
- Give an example of a rearrangement of the series $\sum \frac{(-1)^{n+1}}{n} = \log 2$ which sums to a different value.
- Prove that if $\sum a_n$ is a series with positive terms, and $(b_n)$ is a rearrangement of $(a_n)$ then $\sum b_n = \sum a_n$.
- Prove that if $\sum a_n$ is an absolutely convergent series, and $(b_n)$ is a rearrangement of $(a_n)$ then $\sum b_n = \sum a_n$.
- Conclude that conditionally convergent series are the *only* convergent series whose sum can be altered by rearrangement.
- Know that if $\sum a_n$ is a conditionally convergent series, then for every real number $s$ there is a rearrangement $(a_n)$ of $(a_n)$ such that $\sum b_n = s$.

# Index