Chapter 2

Sequences I

2.1 Introduction

A sequence is a list of numbers in a definite order so that we know which number is in the first place, which number is in the second place and, for any natural number n, we know which number is in the n^{th} place.

All the sequences in this course are infinite and contain only real numbers. For example:

$$1, 2, 3, 4, 5, \ldots$$
$$-1, 1, -1, 1, -1, \ldots$$
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$
$$\sin(1), \sin(2), \sin(3), \sin(4), \ldots$$

In general we denote a sequence by:

$$(a_n) = a_1, a_2, a_3, a_4, \ldots$$

Notice that for each natural number, n, there is a term a_n in the sequence; thus a sequence can be thought of as a function $a : \mathbb{N} \to \mathbb{R}$ given by $a(n) = a_n$. Sequences, like many functions, can be plotted on a graph. Let's denote the first three sequences above by (a_n), (b_n) and (c_n), so the n^{th}terms are given by:

$$\begin{aligned} a_n &= n; \\ b_n &= (-1)^n; \\ c_n &= \frac{1}{n}. \end{aligned}$$

Figure 2.1 shows roughly what the graphs look like.

Another representation is obtained by simply labelling the points of the sequence on the real line, see figure 2.2. These pictures show types of behaviour that a sequence might have. The sequence (a_n) "goes to infinity", the sequence (b_n) "jumps back and forth between -1 and 1", and the sequence (c_n) "converges to 0". In this chapter we will decide how to give each of these phrases a precise meaning.

Initially

Sometimes you will see a_0 as the initial term of a sequence. We will see later that, as far as convergence is concerned, it doesn't matter where you start the sequence.

Sine Time

What do you think the fourth sequence, $\sin(n)$, looks like when you plot it on the real line?

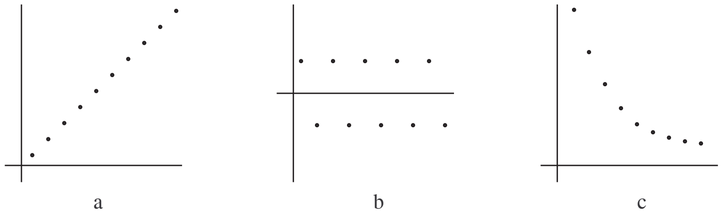

Figure 2.1: Graphing sequences as functions $\mathbb{N} \to \mathbb{R}$.

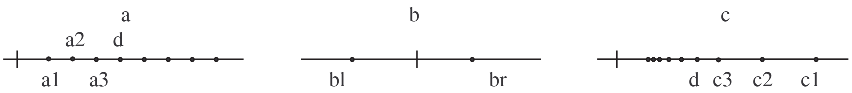

Figure 2.2: Number line representations of the sequences in figure 2.1.

Exercise 1 Write down a formula for the n^{th}term of each of the sequences below. Then plot the sequence in each of the two ways described above.

1. $1, 3, 5, 7, 9, \ldots$
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$
3. $0, -2, 0, -2, 0, -2, \ldots$
4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

2.2 Increasing and Decreasing Sequences

Labour Savers

Note that:
strictly increasing $\Longrightarrow$ increasing (and not decreasing)
strictly decreasing $\Longrightarrow$ decreasing (and not increasing)
increasing $\Longrightarrow$ monotonic
decreasing $\Longrightarrow$ monotonic.

Definition
A sequence (a_n) is:

strictly increasing	if, for all n, $a_n < a_{n+1}$;
increasing	if, for all n, $a_n \leq a_{n+1}$;
strictly decreasing	if, for all n, $a_n > a_{n+1}$;
decreasing	if, for all n, $a_n \geq a_{n+1}$;
monotonic	if it is increasing *or* decreasing *or* both;
non-monotonic	if it is neither increasing *nor* decreasing.

Example Recall the sequences (a_n), (b_n) and (c_n), given by $a_n = n$, $b_n = (-1)^n$ and $c_n = \frac{1}{n}$. We see that:

1. for all n, $a_n = n < n + 1 = a_{n+1}$, therefore (a_n) is strictly increasing;
2. $b_1 = -1 < 1 = b_2$, $b_2 = 1 > -1 = b_3$, therefore (b_n) is neither increasing nor decreasing, i.e. non-monotonic;
3. for all n, $c_n = \frac{1}{n} > \frac{1}{n+1} = c_{n+1}$, therefore (c_n) is strictly decreasing.

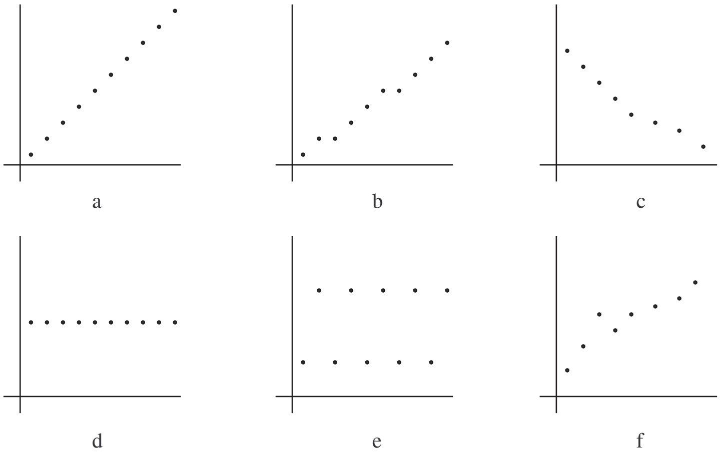

Figure 2.3: Which sequences are monotonic?

Exercise 2 Test whether each of the sequences defined below has any of the following properties: increasing; strictly increasing; decreasing; strictly decreasing; non-monotonic. [A graph of the sequence may help you to decide, but use the formal definitions in your proof.]

1. $a_n = -\frac{1}{n}$	2. $a_{2n-1} = n, a_{2n} = n$	3. $a_n = 1$
4. $a_n = 2^{-n}$	5. $a_n = \sqrt{n+1} - \sqrt{n}$	6. $a_n = \sin n$

Hint: In part 5, try using the identity $a - b = \frac{a^2-b^2}{a+b}$.

Be Dotty

When you are graphing your sequences, remember not to "join the dots". Sequences are functions defined on the *natural numbers* only.

2.3 Bounded Sequences

Definition
A sequence (a_n) is:

bounded above if there exists U such that, for all n, $a_n \leq U$;
U is an *upper bound* for (a_n);
bounded below if there exists L such that, for all n, $a_n \geq L$;
L is a *lower bound* for (a_n);
bounded if it is both bounded above *and* bounded below.

Boundless Bounds

If U is an upper bound then so is any number greater than U. If L is a lower bound then so is any number less than L. *Bounds are not unique.*

Example

1. The sequence $\left(\frac{1}{n}\right)$ is bounded since $0 < \frac{1}{n} \leq 1$.
2. The sequence (n) is bounded below but is not bounded above because for each value C there exists a number n such that $n > C$.

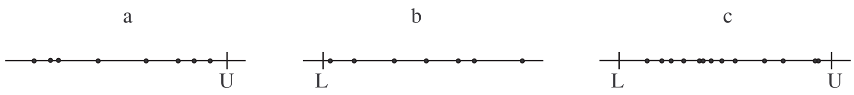

Figure 2.4: Sequences bounded above, below and both.

Bounds for Monotonic Sequences

Each increasing sequence (a_n) is bounded *below* by a_1.
Each decreasing sequence (a_n) is bounded *above* by a_1.

Exercise 3 Decide whether each of the sequences defined below is bounded above, bounded below, bounded. If it is none of these things then explain why. Identify upper and lower bounds in the cases where they exist. Note that, for a positive real number x, $\sqrt{x}$, denotes the positive square root of x.

1.	$\frac{(-1)^n}{n}$	2.	$\sqrt{n}$	3.	$a_n = 1$
4.	$\sin n$	5.	$\sqrt{n+1} - \sqrt{n}$	6.	$(-1)^n n$

Exercise 4

1. A sequence (a_n) is known to be increasing.
   (a) Might it have an upper bound?
   (b) Might it have a lower bound?
   (c) Must it have an upper bound?
   (d) Must is have a lower bound?

   Give a numerical example to illustrate each possibility or impossibility.

2. If a sequence is not bounded above, must it contain
   (a) a positive term,
   (b) an infinite number of positive terms?

2.4 Sequences Tending to Infinity

We say a sequence tends to infinity if its terms eventually exceed any number we choose.

Definition
A sequence (a_n) *tends to infinity* if, for every $C > 0$, there exists a natural number N such that $a_n > C$ for all $n > N$.

We will use three different ways to write that a sequence (a_n) tends to infinity, $(a_n) \to \infty$, $a_n \to \infty$, as $n \to \infty$ and $\lim_{n\to\infty} a_n = \infty$.

Example

1. $\left(\frac{n}{3}\right) \to \infty$. Let $C > 0$. We want to find N such that if $n > N$ then $\frac{n}{3} > C$. Note that $\frac{n}{3} > C \Leftrightarrow n > 3C$. So choose $N \geq 3C$. If $n > N$ then $\frac{n}{3} > \frac{N}{3} \geq C$. In the margin draw a graph of the sequence and illustrate the positions of C and N.

2. $(2^n) \to \infty$. Let $C > 0$. We want to find N such that if $n > N$ then $2^n > C$.

   Note that $2^n > C \Leftrightarrow n > \log_2 C$. So choose $N \geq \log_2 C$. If $n > N$ then $2^n > 2^N \geq 2^{\log_2 C} = C$.

Exercise 5 When does the sequence $(\sqrt{n})$ eventually exceed 2, 12 and 1000? Then prove that $(\sqrt{n})$ tends to infinity.

Exercise 6 Select values of C to demonstrate that the following sequences do not tend to infinity.

1. $1,\ 1,\ 2,\ 1,\ 3,\ \ldots,\ n,\ 1,\ \ldots$
2. $-1,\ 2,\ -3,\ 4,\ \ldots,\ (-1)^n n,\ \ldots$
3. $11,\ 12,\ 11,\ 12,\ \ldots,\ 11,\ 12,\ \ldots$

Is Infinity a Number?

We have not defined "infinity" to be any sort of number - in fact, we have not defined infinity at all. We have side-stepped any need for this by defining the phrase "tends to infinity" as a self-contained unit.

Exercise 7 Think of examples to show that:

1. an increasing sequence need not tend to infinity;
2. a sequence that tends to infinity need not be increasing;
3. a sequence with no upper bound need not tend to infinity.

Theorem
Let (a_n) and (b_n) be two sequences such that $b_n \geq a_n$ for all n. If $(a_n) \to \infty$ then $(b_n) \to \infty$.

Proof. Suppose $C > 0$. We know that there exists N such that $a_n > C$ whenever $n > N$. Hence $b_n \geq a_n > C$ whenever $n > N$. ■

Example We know that $n^2 \geq n$ and $(n) \to \infty$, hence $(n^2) \to \infty$.

Definition
A sequence (a_n) *tends to minus infinity* if, for every $C < 0$, there exists a number N such that $a_n < C$ for all $n > N$.

The corresponding three ways to write that (a_n) tends to minus infinity are $(a_n) \to -\infty$,$a_n \to -\infty$, *as* $n \to \infty$ and $\lim_{n\to\infty} a_n = -\infty$

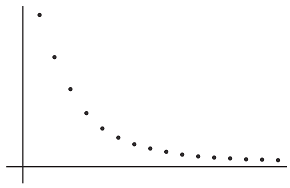

Figure 2.5: Does this look like a null sequence?

Example You can show that $(a_n) \to -\infty$ if and only if $(-a_n) \to \infty$. Hence, $(-n)$, $\left(\frac{-n}{2}\right)$ and $(-\sqrt{n})$ all tend to minus infinity.

Theorem
Suppose $(a_n) \to \infty$ and $(b_n) \to \infty$. Then $(a_n + b_n) \to \infty$, $(a_n b_n) \to \infty$, $(ca_n) \to \infty$ when $c > 0$ and $(ca_n) \to -\infty$ when $c < 0$.

Proof. We'll just do the first part here. Suppose $(a_n) \to \infty$ and $(b_n) \to \infty$. Let $C > 0$. Since $(a_n) \to \infty$ and $C/2 > 0$ there exists a natural number N_1 such that $a_n > C/2$ whenever $n > N_1$. Also, since $(b_n) \to \infty$ and $C/2 > 0$ there exists a natural number N_2 such that $b_n > C/2$ whenever $n > N_2$. Now let $N = \max\{N_1, N_2\}$. Suppose $n > N$. Then

$$n > N_1 \text{ and } n > N_2 \text{ so that } a_n > C/2 \text{ and } b_n > C/2.$$

This gives that

$$a_n + b_n > C/2 + C/2 = C.$$

This is exactly what it means to say that $(a_n + b_n) \to \infty$.

Try doing the other parts in your portfolio. [Hint: for the second part use $\sqrt{C}$ instead of $C/2$ in a proof similar to the above.] ■

2.5 Null Sequences

If someone asked you whether the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$$

"tends to zero", you might draw a graph like figure 2.5 and then probably answer "yes". After a little thought you might go on to say that the sequences

$$1, 0, \frac{1}{2}, 0, \frac{1}{3}, , 0, \frac{1}{4}, 0, \ldots, \frac{1}{n}, 0, \ldots$$

and

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, (-1)^n \frac{1}{n}, \ldots$$

also "tend to zero".

Is Zero Allowed?

We are going to allow zeros to appear in sequences that "tend to zero" and not let their presence bother us. We are even going to say that the sequence

$$0, 0, 0, 0, 0, \ldots$$

"tends to zero".

We want to develop a precise definition of what it means for a sequence to "tend to zero". As a first step, notice that for each of the sequences above, every positive number is eventually an upper bound for the sequence and every negative number is eventually a lower bound. (So the sequence is getting "squashed" closer to zero the further along you go.)

Exercise 8

1. Use the sequences below (which are not null) to demonstrate the inadequacy of the following attempts to define a null sequence.

   (a) A sequence in which each term is strictly less than its predecessor.

   (b) A sequence in which each term is strictly less than its predecessor while remaining positive.

   (c) A sequence in which, for sufficiently large n, each term is less than some small positive number.

   (d) A sequence in which, for sufficiently large n, the absolute value of each term is less than some small positive number.

   (e) A sequence with arbitrarily small terms.

   I. $2,\ 1,\ 0,\ -1,\ -2,\ -3,\ -4,\ \ldots,\ -n,\ \ldots$

   II. $2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \frac{6}{5},\ \ldots,\ \frac{n+1}{n},\ \ldots$

   III. $2,\ 1,\ 0,\ -1,\ -0.1,\ -0.1,\ -0.1,\ \ldots,\ -0.1,\ \ldots$

   IV. $2,\ 1,\ 0,\ -0.1,\ 0.01,\ -0.001,\ 0.01,\ -0.001,\ \ldots,\ 0.01,\ -0.001,\ \ldots$

   V. $1,\ \frac{1}{2},\ 1,\ \frac{1}{4},\ 1,\ \frac{1}{8},\ \ldots$

2. Examine the sequence

$$-1,\ \frac{1}{2},\ -\frac{1}{3},\ \frac{1}{4},\ -\frac{1}{5},\ \ldots,\ \frac{(-1)^n}{n},\ \ldots$$

   (a) Beyond what stage in the sequence are the terms between -0.1 and 0.1?

   (b) Beyond what stage in the sequence are the terms between -0.01 and 0.01?

   (c) Beyond what stage in the sequence are the terms between -0.001 and 0.001?

   (d) Beyond what stage in the sequence are the are the terms between $-\varepsilon$ and ε, where ε is a given positive number?

εrror.
The choice of ε, the Greek e, is to stand for 'error', where the terms of a sequence are thought of as sucessive attempts to hit the target of 0.

You noticed in Exercise 8 (2.) that for every value of ε, no matter how tiny, the sequence was eventually sandwiched between ε and $-\varepsilon$ (i.e. within ε of zero). We use this observation to create our definition. See figure 2.6

Definition
A sequence (a_n) *tends to zero* if, for each $\varepsilon > 0$, there exists a natural number N such that $|a_n| < \varepsilon$ for all $n > N$.

Make Like an Elephant

This definition is the trickiest we've had so far. Even if you don't understand it yet
Memorise It!
In fact, memorise all the other definitions while you're at it.

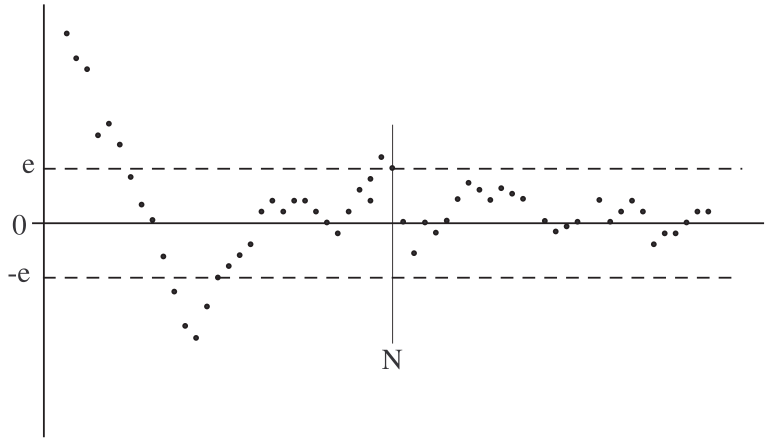

Figure 2.6: Null sequences; first choose ε, then find N.

The three ways to write a sequence tends to zero are, $(a_n) \to 0$, $a_n \to 0$, as $n \to \infty$, and $\lim_{n\to\infty} a_n = 0$. We also say (a_n) *converges to zero*, or (a_n) is a *null sequence.*

Archimedean Property
One property of the real numbers that we don't often give much thought to is this:
Given any real number x there is an integer N such that $N > x$.
Where have we used this fact?

Example The sequence $(a_n) = \left(\frac{1}{n}\right)$ tends to zero. Let $\varepsilon > 0$. We want to find N such that if $n > N$, then $|a_n| = \frac{1}{n} < \varepsilon$.

Note that $\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$. So choose a natural number $N \geq \frac{1}{\varepsilon}$. If $n > N$, then $|a_n| = \frac{1}{n} < \frac{1}{N} \leq \varepsilon$.

Exercise 9 Prove that the sequence $\left(\frac{1}{\sqrt{n}}\right)$ tends to zero.

Exercise 10 Prove that the sequence $(1, 1, 1, 1, 1, 1, 1, \dots)$ does *not* tend to zero. (Find a value of ε for which there is no corresponding N.)

Lemma
If $(a_n) \to \infty$ and $a_n \neq 0$ for any n, then $\left(\frac{1}{a_n}\right) \to 0$.

Exercise 11 Prove this lemma.

Exercise 12 Think of an example to show that the opposite statement,

$$\text{if } (a_n) \to 0 \text{ then } \left(\frac{1}{a_n}\right) \to \infty,$$

is *false*, even if $a_n \neq 0$ for all n.

Lemma *Absolute Value Rule*
$(a_n) \to 0$ if and only if $(|a_n|) \to 0$.

Proof. The absolute value of $|a_n|$ is just $|a_n|$, i.e.$||a_n|| = |a_n|$. So $|a_n| \to 0$ iff for each $\varepsilon > 0$ there exists a natural number N such that $|a_n| < \varepsilon$ whenever $n > N$. But, by definition, this is exactly what $(a_n) \to 0$ means. ■

Example We showed before that $\left(\frac{1}{n}\right) \to 0$. Now $\frac{1}{n} = \left|\frac{(-1)^n}{n}\right|$. Hence $\left(\frac{(-1)^n}{n}\right) \to 0$.

Theorem *Sandwich Theorem for Null Sequences*
Suppose $(a_n) \to 0$. If $|b_n| \le |a_n|$ then $(b_n) \to 0$.

Example

1. Clearly $0 \le \frac{1}{n+1} \le \frac{1}{n}$. Therefore $\left(\frac{1}{n+1}\right) \to 0$.
2. $0 \le \frac{1}{n^{3/2}} \le \frac{1}{n}$. Therefore $\left(\frac{1}{n^{3/2}}\right) \to 0$.

Exercise 13 Prove that the following sequences are null using the result above. Indicate what null sequence you are using to make your Sandwich.

$$1.\ \left(\frac{\sin n}{n}\right) \qquad 2.\ \left(\sqrt{n+1} - \sqrt{n}\right)$$

2.6 Arithmetic of Null Sequences

Theorem
Suppose $(a_n) \to 0$ and $(b_n) \to 0$. Then for all numbers c and d:

$$\begin{aligned} (ca_n + db_n) &\to 0 && \textit{Sum Rule} \text{ for Null Sequences;} \\ (a_n b_n) &\to 0 && \textit{Product Rule} \text{ for Null Sequences.} \end{aligned}$$

Examples

- $\left(\frac{1}{n^2}\right) = \left(\frac{1}{n} \cdot \frac{1}{n}\right) \to 0$ (Product Rule)
- $\left(\frac{2n-5}{n^2}\right) = \left(\frac{2}{n} - \frac{5}{n^2}\right) \to 0$ (Sum Rule)

The Sum Rule and Product Rule are hardly surprising. If they failed we would surely have the wrong definition of a null sequence. So proving them carefully acts as a test to see if our definition is working.

Exercise 14

1. If (a_n) is a null sequence and c is a constant number, prove that $(c \cdot a_n)$ is a null sequence. [Hint: Consider the cases $c \neq 0$ and $c = 0$ in turn].
2. Deduce that $\frac{10}{\sqrt{n}}$ is a null sequence.
3. Suppose that (a_n) and (b_n) are both null sequences, and $\varepsilon > 0$ is given.
   (a) Must there be an N_1 such that $|a_n| < \frac{1}{2}\varepsilon$ when $n > N_1$?
   (b) Must there be an N_2 such that $|b_n| < \frac{1}{2}\varepsilon$ when $n > N_2$?
   (c) Is there an N_0 such that when $n > N_0$ both $n > N_1$ and $n > N_2$?
   (d) If $n > N_0$ must $|a_n + b_n| < \varepsilon$?
   You have proved that the termwise *sum* of two null sequences is null.
   (e) If the sequence (c_n) is also null, what about $(a_n + b_n + c_n)$? What about the sum of k null sequences?

Exercise 15 Suppose (a_n) and (b_n) are both null sequences, and $\varepsilon > 0$ is given.

1. Must there be an N_1 such that $|a_n| < \varepsilon$ when $n > N_1$?
2. Must there be an N_2 such that $|b_n| < 1$ when $n > N_2$?
3. Is there an N_0 such that when $n > N_0$ both $n > N_1$ and $n > N_2$?
4. If $n > N_0$ must $|a_n b_n| < \varepsilon$?
   You have proved that the termwise *product* of two null sequences is null.
5. If the sequence (c_n) is also null, what about $(a_n b_n c_n)$? What about the product of k null sequences?

Example To show that $\left(\frac{n^2+2n+3}{n^3}\right)$ is a null sequence, note that $\frac{n^2+2n+3}{n^3} = \frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3}$. We know that $\left(\frac{1}{n}\right) \to 0$ so $\left(\frac{1}{n^2}\right)$ and $\left(\frac{1}{n^3}\right)$ are null by the Product Rule. It follows that $\left(\frac{n^2+2n+3}{n^3}\right)$ is null by the Sum Rule.

2.7 * Application - Estimating π *

Recall Archimedes' method for approximating π: A_n and a_n are the areas of the circumscribed and inscribed regular n sided polygon to a circle of radius 1. Archimedes used the formulae

$$a_{2n} = \sqrt{a_n A_n} \qquad A_{2n} = \frac{2A_n a_{2n}}{A_n + a_{2n}}$$

to estimate π.

Exercise 16 Why is the sequence $a_4, a_8, a_{16}, a_{32}, \ldots$ increasing? Why are all the values between 2 and π? What similar statements can you make about the sequence $A_4, A_8, A_{16}, A_{32}, \ldots$?

Using Archimedes' formulae we see that

$$\begin{aligned}
A_{2n} - a_{2n} &= \frac{2A_n a_{2n}}{A_n + a_{2n}} - a_{2n} \\
&= \frac{A_n a_{2n} - a_{2n}^2}{A_n + a_{2n}} \\
&= \frac{a_{2n}}{A_n + a_{2n}} (A_n - a_{2n}) \\
&= \frac{a_{2n}}{A_n + a_{2n}} \left(A_n - \sqrt{a_n A_n}\right) \\
&= \left(\frac{a_{2n}\sqrt{A_n}}{(A_n + a_{2n})(\sqrt{A_n} + \sqrt{a_n})}\right) (A_n - a_n)
\end{aligned}$$

Exercise 17 Explain why $\left(\frac{a_{2n}\sqrt{A_n}}{(A_n+a_{2n})(\sqrt{A_n}+\sqrt{a_n})}\right)$ is never larger than 0.4. [Hint: use the bounds from the previous question.] Hence show that the error $(A_n - a_n)$ in calculating π reduces by at least 0.4 when replacing n by $2n$. Show that by calculating $A_{2^{10}}$ and $a_{2^{10}}$ we can estimate π to within 0.0014. [Hint: recall that $a_n \leq \pi \leq A_n$.]

Check Your Progress
By the end of this chapter you should be able to:

- Explain the term "sequence" and give a range of examples.
- Plot sequences in two different ways.
- Test whether a sequence is (strictly) increasing, (strictly) decreasing, monotonic, bounded above or bounded below - and formally state the meaning of each of these terms.
- Test whether a sequence "tends to infinity" and formally state what that means.
- Test whether a sequence "tends to zero" and formally state what that means.
- Apply the Sandwich Theorem for Null Sequences.
- Prove that if (a_n) and (b_n) are null sequences then so are $(|a_n|)$, (ca_n+db_n) and (a_nb_n).