Chapter 3

Sequences II

3.1 Convergent Sequences

Plot a graph of the sequence \((a_n) = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots\). To what limit do you think this sequence tends? What can you say about the sequence \((a_n - 1)\)?

For \(\epsilon = 0.1, \epsilon = 0.01\) and \(\epsilon = 0.001\) find an \(N\) such that \(|a_n - 1| < \epsilon\) whenever \(n > N\).

See figure 3.1 for an illustration of this definition.

We use the notation \((a_n) \to a, a_n \to a, \) as \(n \to \infty\) and \(\lim_{n \to \infty} a_n = a\) and say that \((a_n)\) converges to \(a\), or the limit of the sequence \((a_n)\) as \(n\) tends to infinity is \(a\).

**Example**  Prove \((a_n) = \left(\frac{n}{n+1}\right) \to 1\).

Let \(\epsilon > 0\). We have to find a natural number \(N\) so that

\[|a_n - 1| = \left|\frac{n}{n+1} - 1\right| < \epsilon\]

when \(n > N\). We have

\[\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n}\]

Hence it suffices to find \(N\) so that \(\frac{1}{n} < \epsilon\) whenever \(n > N\). But \(\frac{1}{n} < \epsilon\) if and only if \(\frac{1}{\epsilon} < n\) so it is enough to choose \(N\) to be a natural number with \(N > \frac{1}{\epsilon}\). Then, if \(n > N\) we have

\[|a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} < \epsilon.\]

**Lemma**  \((a_n) \to a\) if and only if \((a_n - a) \to 0\).

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**Good N-ough**

Any \(N\) that works is good enough - it doesn’t have to be the smallest possible \(N\).

**Recycle**

Have a closer look at figure 3.1, what has been changed from figure 2.6 of chapter 2? It turns out that this definition is very similar to the definition of a null sequence.

**Elephants Revisited**

A null sequence is a special case of a convergent sequence. So memorise this definition and get the other one for free.
Proof. We know that \((a_n - a) \to 0\) means that for each \(\epsilon > 0\), there exists a natural number \(N\) such that \(|a_n - a| < \epsilon\) when \(n > N\). But this is exactly the definition of \((a_n) \to a\). \(\blacksquare\)

We have spoken of the limit of a sequence but can a sequence have more than one limit? The answer had better be “No” or our definition is suspect.

**Theorem** Uniqueness of Limits

A sequence cannot converge to more than one limit.

**Exercise 1** Prove the theorem by assuming \((a_n) \to a\), \((a_n) \to b\) with \(a < b\) and obtaining a contradiction. [Hint: try drawing a graph of the sequences with \(a\) and \(b\) marked on]

**Theorem**

Every convergent sequence is bounded.

**Exercise 2** Prove the theorem above.

### 3.2 “Algebra” of Limits

**Connection**

It won’t have escaped your notice that the Sum Rule for null sequences is just a special case of the Sum Rule for sequences. The same goes for the Product Rule.

Why don’t we have a Quotient Rule for null sequences?

**Theorem** Sum Rule, Product Rule and Quotient Rule

Let \(a, b \in \mathbb{R}\). Suppose \((a_n) \to a\) and \((b_n) \to b\). Then

\[
(ca_n + db_n) \to ca + db \quad \text{Sum Rule for Sequences}
\]

\[
(a_nb_n) \to ab \quad \text{Product Rule for Sequences}
\]

\[
\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}, \quad \text{if } b \neq 0 \quad \text{Quotient Rule for Sequences}
\]
There is another useful way we can express all these rules: If \((a_n)\) and \((b_n)\) are convergent then

\[
\lim_{n \to \infty} (ca_n + db_n) = c \lim_{n \to \infty} a_n + d \lim_{n \to \infty} b_n \quad \text{Sum Rule}
\]
\[
\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n \quad \text{Product Rule}
\]
\[
\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \frac{\lim_{n \to \infty} (a_n)}{\lim_{n \to \infty} (b_n)}, \text{ if } \lim_{n \to \infty} (b_n) \neq 0 \quad \text{Quotient Rule}
\]

**Example** In full detail

\[
\lim_{n \to \infty} \left( \frac{n^2 + 1}{2n^3 + 5} \right) \cdot (6n - 1) = \lim_{n \to \infty} \left( \frac{1 + \frac{1}{n^2}}{2 + \frac{5}{n}} \right) \cdot \left( \frac{6 - \frac{1}{n}}{6 - \frac{1}{n}} \right)
\]

using the Quotient Rule

\[
= \lim_{n \to \infty} \left( \frac{1 + \frac{1}{n^2}}{2} \right) \cdot \left( \frac{6 - \frac{1}{n}}{6 - \frac{1}{n}} \right)
\]

using the Product and Sum Rules

\[
= \frac{(1 + 0)(6 - 0)}{2 + 5 \lim_{n \to \infty} \left( \frac{1}{n^3} \right)}
\]

\[
= \frac{6}{2 + 0} = 3
\]

Unless you are asked to show where you use each of the rules you can keep your solutions simpler. Either of the following is fine:

\[
\lim_{n \to \infty} \left( \frac{n^2 + 1}{2n^3 + 5} \right) \cdot (6n - 1) = \lim_{n \to \infty} \left( \frac{1 + \frac{1}{n^2}}{2 + \frac{5}{n}} \right) \cdot \left( \frac{6 - \frac{1}{n}}{6 - \frac{1}{n}} \right) \cdot (6 - 0) = 3
\]

or

\[
\lim_{n \to \infty} \left( \frac{n^2 + 1}{2n^3 + 5} \right) \cdot (6n - 1) = \frac{(1 + \frac{1}{n^2})}{2 + \frac{5}{n}} \cdot \frac{(6 - \frac{1}{n})}{6 - \frac{1}{n}} \cdot (6 - 0) = 3
\]

**Exercise 3** Show that

\[(a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a) = a_nb_n - ab\]

**Exercise 4** Use the identity in Exercise 3 and the rules for null sequences to prove the Product Rule for sequences.

**Exercise 5** Write a proof of the Quotient Rule. You might like to structure your proof as follows.

1. Note that \((bb_n) \to b^2\) and show that \(bb_n > \frac{b^2}{2}\) for sufficiently large \(n\).
2. Then show that eventually $0 \leq \left| \frac{1}{b_n} - \frac{1}{b} \right| \leq \frac{2}{5} |b-b_n|$ and therefore $\left( \frac{1}{b_n} \right) \to \frac{1}{b}$.

3. Now tackle $\frac{a_n}{b_n} = a_n \frac{1}{b_n}$.

Exercise 6 Find the limit of each of the sequences defined below.

1. $\frac{7n^2 + 8}{4n^2 - 5n}$
2. $\frac{2n + 1}{2n - 1}$
3. $\frac{3n^2}{\sqrt{n} + 3} \left( \sqrt{n} - 2 \right)$
4. $\frac{1 + 2 + \cdots + n}{n^2}$

3.3 Further Useful Results

The Sandwich Rule for null sequences represents the case when $l = 0$.

Theorem Sandwich Theorem for Sequences

Suppose $(a_n) \to l$ and $(b_n) \to l$. If $a_n \leq c_n \leq b_n$, then $(c_n) \to l$.

This improved Sandwich theorem can be tackled by rewriting the hypothesis as $0 \leq c_n - a_n \leq b_n - a_n$ and applying the earlier Sandwich theorem.

Exercise 7 Prove the Sandwich Theorem for sequences.

There are going to be many occasions when we are interested in the behaviour of a sequence “after a certain point”, regardless of what went on before that. This can be done by “chopping off” the first $N$ terms of a sequence $(a_n)$ to get a shifted sequence $(b_n)$ given by $b_n = a_{N+n}$. We often write this as $(a_{N+n})$, so that

$$(a_{N+n}) = a_{N+1}, \ a_{N+2}, \ a_{N+3}, \ a_{N+4}, \ \cdots$$

which starts at the term $a_{N+1}$. We use it in the definition below.

Definition A sequence $(a_n)$ satisfies a certain property eventually if there is a natural number $N$ such that the sequence $(a_{N+n})$ satisfies that property.

For instance, a sequence $(a_n)$ is eventually bounded if there exists $N$ such that the sequence $(a_{N+n})$ is bounded.

Lemma If a sequence is eventually bounded then it is bounded.

Exercise 8 Prove this lemma.

The next result, called the Shift Rule, tells you that a sequence converges if and only if it converges eventually. So you can chop off or add on any finite
number of terms at the beginning of a sequence without affecting the convergent behaviour of its infinite “tail”.

**Theorem** Shift Rule
Let \( N \) be a natural number. Let \((a_n)\) be a sequence. Then \(a_n \to a\) if and only if the “shifted” sequence \(a_{N+n} \to a\).

**Proof.** Fix \( \epsilon > 0 \). If \((a_n) \to a\) we know there exists \( N_1 \) such that \(|a_n - a| < \epsilon\) whenever \( n > N_1 \). When \( n > N_1 \), we see that \( N+n > N_1 \), therefore \(|a_{N+n} - a| < \epsilon\). Hence \((a_{N+n}) \to a\). Conversely, suppose that \((a_{N+n}) \to a\). Then there exists \( N_2 \) such that \(|a_{N+n} - a| < \epsilon\) whenever \( n > N_2 \). When \( n > N + N_2 \) then \( n - N > N_2 \) so \(|a_n - a| = |a_{N+(n-N)} - a| < \epsilon\). Hence \((a_n) \to a\). \(\blacksquare\)

**Corollary** Sandwich Theorem with Shift Rule
Suppose \((a_n) \to l\) and \((b_n) \to l\). If eventually \(a_n \leq c_n \leq b_n\) then \((c_n) \to l\).

**Example** We know \(1/n \to 0\) therefore \(1/(n+5) \to 0\).

**Exercise 9** Show that the Shift Rule also works for sequences which tend to infinity: \((a_n) \to \infty\) if and only if \((a_{N+n}) \to \infty\).

If all the terms of a convergent sequence sit within a certain interval, does its limit lie in that interval, or can it “escape”? For instance, if the terms of a convergent sequence are all positive, is its limit positive too?

**Lemma**
Suppose \((a_n) \to a\). If \(a_n \geq 0\) for all \(n\) then \(a \geq 0\).

**Exercise 10** Prove this result. [Hint: Assume that \(a < 0\) and let \(\epsilon = -a > 0\). Then use the definition of convergence to arrive at a contradiction.]

**Exercise 11** Prove or disprove the following statement:

“Suppose \((a_n) \to a\). If \(a_n > 0\) for all \(n\) then \(a > 0\).”

**Theorem** Inequality Rule
Suppose \((a_n) \to a\) and \((b_n) \to b\). If (eventually) \(a_n \leq b_n\) then \(a \leq b\).

**Exercise 12** Prove this result using the previous Lemma. [Hint: Consider \((b_n - a_n)\).]
CHAPTER 3. SEQUENCES II

Limits on Limits
Limits cannot escape from closed intervals. They can escape from open intervals - but only as far as the end points.

Corollary  Closed Interval Rule
Suppose \((a_n) \to a\). If (eventually) \(A \leq a_n \leq B\) then \(A \leq a \leq B\).

If \(A < a_n < B\) it is not the case that \(A < a < B\). For example \(0 < \frac{n}{n+1} < 1\) but \(\frac{n}{n+1} \to 1\).

3.4 Subsequences

A subsequence of \((a_n)\) is a sequence consisting of some (or all) of its terms in their original order. For instance, we can pick out the terms with even index to get the subsequence \(a_2, a_4, a_6, a_8, a_{10}, \ldots\)
or we can choose all those whose index is a perfect square \(a_1, a_4, a_9, a_{16}, a_{25}, \ldots\)

In the first case we chose the terms in positions 2, 4, 6, 8, \ldots and in the second those in positions 1, 4, 9, 16, 25, \ldots

In general, if we take any strictly increasing sequence of natural numbers \((n_i) = n_1, n_2, n_3, n_4, \ldots\) we can define a subsequence of \((a_n)\) by \((a_{n_i}) = a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots\)

**Definition**

A subsequence of \((a_n)\) is a sequence of the form \((a_{n_i})\), where \((n_i)\) is a strictly increasing sequence of natural numbers.

Effectively, the sequence \((n_i)\) “picks out” which terms of \((a_n)\) get to belong to the subsequence. Think back to the definition of convergence of a sequence. Why is it immediate from the definition that if a sequence \((a_n)\) converges to \(a\) then all subsequence \((a_{n_i})\) converge to \(a\)? This is a fact which we will be using constantly in the rest of the course.

Notice that the shifted sequence \((a_{N+n})\) is a subsequence of \((a_n)\).

**Exercise 13** Let \((a_n) = (n^2)\). Write down the first four terms of the three subsequences \((a_{n+4})\), \((a_{3n-1})\) and \((a_{2n})\).

Here is another result which we will need in later chapters.

**Exercise 14** Suppose we have a sequence \((a_n)\) and are trying to prove that it converges. Assume that we have shown that the subsequences \((a_{2n})\) and \((a_{2n+1})\) both converge to the same limit \(a\). Prove that \((a_n) \to a\) converges.

**Exercise 15** Answer “Yes” or “No” to the following questions, but be sure that you know why and that you aren’t just guessing.
1. A sequence \((a_n)\) is known to be increasing, but not strictly increasing.
   (a) Might there be a strictly increasing subsequence of \((a_n)\)?
   (b) Must there be a strictly increasing subsequence of \((a_n)\)?

2. If a sequence is bounded, must every subsequence be bounded?

3. If the subsequence \(a_2, a_3, \ldots, a_{n+1}, \ldots\) is bounded, does it follow that the sequence \((a_n)\) is bounded?

4. If the subsequence \(a_3, a_4, \ldots, a_{n+2}, \ldots\) is bounded does it follow that the sequence \((a_n)\) is bounded?

5. If the subsequence \(a_{N+1}, a_{N+2}, \ldots, a_{N+n}, \ldots\) is bounded does it follow that the sequence \((a_n)\) is bounded?

**Lemma**

Every subsequence of a bounded sequence is bounded.

**Proof.** Let \((a_n)\) be a bounded sequence. Then there exist \(L\) and \(U\) such that \(L \leq a_n \leq U\) for all \(n\). It follows that if \((a_{n_i})\) is a subsequence of \((a_n)\) then \(L \leq a_{n_i} \leq U\) for all \(i\). Hence \((a_{n_i})\) is bounded.

You might be surprised to learn that every sequence, no matter how bouncy and ill-behaved, contains an increasing or decreasing subsequence.

**Theorem**

Every sequence has a monotonic subsequence.

### 3.5 * Application - Speed of Convergence *

Often sequences are defined **recursively**, that is, later terms are defined in terms of earlier ones. Consider a sequence \((a_n)\) where \(a_0 = 1\) and \(a_{n+1} = \sqrt{a_n + 2}\), so the sequence begins \(a_0 = 1, a_1 = \sqrt{3}, a_2 = \sqrt{\sqrt{3} + 2}\).

**Exercise 16** Use induction to show that \(1 \leq a_n \leq 2\) for all \(n\).

Now assume that \((a_n)\) converges to a limit, say, \(a\). Then:

\[
a = \lim_{n \to \infty} a_n = \lim_{n \to \infty} ((a_{n+1})^2 - 2) = (\lim_{n \to \infty} a_{n+1})^2 - 2 = a^2 - 2
\]

So to find \(a\) we have to solve the quadratic equation \(a^2 - a - 2 = 0\). We can rewrite this as \((a + 1)(a - 2) = 0\), so either \(a = -1\) or \(a = 2\). But which one is it? The Inequality Theorem comes to our rescue here. Since \(a_n \geq 1\) for all \(n\) it follows that \(a \geq 1\), therefore \(a = 2\). We will now investigate the speed that \(a_n\) approaches 2.
Exercise 17 Show that \(2 - a_{n+1} = \frac{2-a_n}{2\sqrt{2} + a_n}\). Use this identity and induction to show that \((2 - a_n) \leq \frac{1}{(2 + \sqrt{2})^n}\) for all \(n\). How many iterations are needed so that \(a_n\) is within \(10^{-100}\) of its limit 2?

An excellent method for calculating square roots is the Newton-Raphson method which you may have met at A-level. When applied to the problem of calculating \(\sqrt{2}\) this leads to the sequence given by: \(a_0 = 2\) and \(a_{n+1} = \frac{1}{a_n} + \frac{a_n}{2}\).

Exercise 18 Use a calculator to calculate \(a_1, a_2, a_3, a_4\). Compare them with \(\sqrt{2}\).

Exercise 19 Use induction to show that \(1 \leq a_n \leq 2\) for all \(n\). Assuming that \((a_n)\) converges, show that the limit must be \(\sqrt{2}\).

We will now show that the sequence converges to \(\sqrt{2}\) like a bat out of hell.

Exercise 20 Show that \((a_{n+1} - \sqrt{2}) = \frac{(a_n - \sqrt{2})^2}{2a_n}\). Using this identity show by induction that \(|a_n - \sqrt{2}| \leq \frac{1}{2^n}\). How many iterations do you need before you can guarantee to calculate \(\sqrt{2}\) to within an error of \(10^{-100}\) (approximately 100 decimal places)?

Sequences as in Exercise 17 are said to converge \textit{exponentially} and those as in Exercise 20 are said to converge \textit{quadratically} since the error is squared at each iteration. The standard methods for calculating \(\pi\) were exponential (just as is the Archimedes method) until the mid 1970s when a quadratically convergent approximation was discovered.

Check Your Progress

By the end of this chapter you should be able to:

- Define what it means for a sequence to “converge to a limit”.
- Prove that every convergent sequence is bounded.
- State, prove and use the following results about convergent sequences: If 
  \((a_n) \to a\) and \((b_n) \to b\) then:
  - **Sum Rule:** \((ca_n + db_n) \to ca + db\)
  - **Product Rule:** \((a_nb_n) \to ab\)
  - **Quotient Rule:** \((a_n/b_n) \to a/b\) if \(b \neq 0\)
  - **Sandwich Theorem:** if \(a = b\) and \(a_n \leq c_n \leq b_n\) then \((c_n) \to a\)
  - **Closed Interval Rule:** if \(A \leq a_n \leq B\) then \(A \leq a \leq B\)
- Explain the term “subsequence” and give a range of examples.