Chapter 5

Completeness I

Completeness is the key property of the real numbers that the rational numbers lack. Before examining this property we explore the rational and irrational numbers, discovering that both sets populate the real line more densely than you might imagine, and that they are inextricably entwined.

5.1 Rational Numbers

Definition
A real number is *rational* if it can be written in the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$. A real number that is not rational is termed *irrational*.

Example $\frac{1}{2}, -\frac{5}{6}, 100, \frac{567877}{-1239}, \frac{8}{2}$ are all rational numbers.

Exercise 1

1. What do you think the letter $\mathbb{Q}$ stands for?
2. Show that each of the following numbers is rational: 0, -10, 2.87, 0.0001, -8^{-9}, 0.6666. . . .
3. Prove that between any two distinct rational numbers there is another rational number.
4. Is there a smallest positive rational number?
5. If a is rational and b is irrational, are $a+b$ and ab rational or irrational? What if a and b are both rational? Or both irrational?

A sensible question to ask at this point is this: are all real numbers rational? In other words, can any number (even a difficult one like π or e) be expressed as a simple fraction if we just try hard enough? For good or ill this is not the case, because, as the Greeks discovered:

Historical Roots
The proof that $\sqrt{2}$ is irrational is attributed to Pythagoras *ca*. $580 - 500$ BC who is well known to have had a triangle fetish.

What does $\sqrt{2}$ have to do with triangles?

Theorem
$\sqrt{2}$ is irrational.

Euler's constant

Euler's γ constant is defined by

$$\gamma = \lim_{n\to\infty}\Big[\sum_{k=1}^{n}\frac{1}{k} - \log n\Big] = 0.5772...$$

It is *not known* whether γ is rational or irrational. It is only known that, if $\gamma = \frac{p}{q}$, q is larger than $10^{10'000}$.

This theorem assures us that *at least one* real number is not rational. You will meet the famous proof of this result in the Foundations course. Later in the course you will prove that e is irrational. The proof that π is irrational is also not hard but somewhat long and you will probably not meet it unless you hunt for it.

We now discover that, despite the fact that some numbers are irrational, the rationals are spread so thickly over the real line that you will find one wherever you look.

Exercise 2

1. Illustrate on a number line those portions of the sets

$$\{m|\, m \in \mathbb{Z}\}, \quad \{m/2|\, m \in \mathbb{Z}\}, \quad \{m/4|\, m \in \mathbb{Z}\}, \quad \{m/8|m \in \mathbb{Z}\}$$

   that lie between ± 3. Is each set contained in the set which follows in the list? What would an illustration of the set $\{m/2^n|\, m \in \mathbb{Z}\}$ look like for some larger positive integer n?

2. Find a rational number which lies between 57/65 and 64/73 and may be written in the form $m/2^n$, where m is an integer and n is a non-negative integer.

Integer Part

If x is a real number then $[x]$, the *integer part* of x, is the unique integer such that

$$[x] \le x < [x] + 1.$$

For example

$$[3.14] = 3 \text{ and } [-3.14] = -4.$$

Open Interval

For $a < b \in \mathbb{R}$, the open interval (a, b) is the set of all numbers *strictly* between a and b: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

Chalk and Cheese

Though the rationals and irrationals share certain properties, do not be fooled into thinking that they are two-of-a-kind. You will learn later that the rationals are "countable", you can pair them up with the natural numbers. The irrationals, however, are manifestly "uncountable"

Theorem
Between any two distinct real numbers there is a rational number.

I.e. if $a < b$, there is a rational $\frac{p}{q}$ with $a < \frac{p}{q} < b$.

Proof. Consider the set of numbers of the form $\frac{p}{q}$ with q fixed, and p any integer. Assume that there are no such numbers between a and b. Let $\frac{p}{q}$ be the number immediately before a. Then $\frac{p+1}{q}$ is the number immediately after b. We necessarily have

$$\frac{p+1}{q} - \frac{p}{q} \ge b - a \quad \Longleftrightarrow \quad \frac{1}{q} \ge b - a.$$

If we choose q sufficiently large, then the above inequality is wrong. Then there is at least one rational number between a and b. ■

Corollary
Let $a < b$. There is an infinite number of rational numbers in the open interval (a, b).

Proof. One can think of many proofs. One could proceed as above, but proving that there are more than N numbers between a and b, for arbitrarily large N. But we can also use the theorem directly. We know that there must be a rational number, say c_1, between a and b. Then there is another rational number, say c_2, between c_1 and b. Then there is c_3, etc... All those numbers are distinct and they are between a and b. ■

We have shown that the rational numbers are spread densely over the real line. What about the irrational numbers?

Exercise 3 Prove that between any two distinct *rational* numbers there is an irrational number. [Hint: Use $\sqrt{2}$ and consider the distance between your two rationals.]

Theorem
Between any two distinct real numbers there is an irrational number.

Proof. We can proceed as in the proof of the previous theorem. Consider the set of numbers of the form $\frac{p}{q} + \sqrt{2}$ with q fixed, and p any integer; all those numbers are irrational. Assume that there are no such numbers between a and b. Let $\frac{p}{q} + \sqrt{2}$ be the number immediately before a. Then $\frac{p+1}{q} + \sqrt{2}$ is the number immediately after b. We necessarily have

$$\frac{p+1}{q} + \sqrt{2} - \left(\frac{p}{q} + \sqrt{2}\right) \geq b - a \quad \Longleftrightarrow \quad \frac{1}{q} \geq b - a.$$

If we choose q sufficiently large, then the above inequality is wrong. Then there is at least one irrational number between a and b. ■

Corollary
Let $a < b$. There is an infinite number of irrational numbers in the open interval (a, b).

Whatever method you used to prove the last corollary will work for this one too. Can you see why?

Is It Love?
We have shown that between any two rationals there is an infinite number of irrationals, and that between any two irrationals there is an infinite number of rationals. So the two sets are intimately and inextricably entwined.

Try to picture the two sets on the real line.

5.2 Least Upper Bounds and Greatest Lower Bounds

Definition
A non-empty set A of real numbers is

bounded above if there exists U such that $a \leq U$ for all $a \in A$; U is an *upper bound* for A.
bounded below if there exists L such that $a \geq L$ for all $a \in A$; L is a *lower bound* for A.
bounded if it is both bounded above *and* below.

Boundless Bounds
If U is an upper bound then so is any number greater than U. If L is a lower bound then so is any number less than L.

Bounds are not unique

Exercise 4 For each of the following sets of real numbers decide whether the set is bounded above, bounded below, bounded or none of these:

1. $\{x : x^2 < 10\}$ 2. $\{x : x^2 > 10\}$ 3. $\{x : x^3 > 10\}$ 4. $\{x : x^3 < 10\}$

Definition
A number u is a *least upper bound* of A if
1. u is an upper bound of A and
2. if U is any upper bound of A then $u \leq U$.

A number l is a *greatest lower bound* of A if
1. l is a lower bound of A and
2. if L is any lower bound of A then $l \geq L$.

The least upper bound of a set A is also called the *supremum* of A and is denoted by $\sup A$, pronounced "soup A".
The greatest lower bound of a set A is also called the *infimum* of A and is denoted by $\inf A$.
Example Let $A = \{\frac{1}{n} : n = 2, 3, 4, \dots\}$. Then $\sup A = 1/2$ and $\inf A = 0$.

Exercise 5 Check that 0 is a lower bound and 2 is an upper bound of each of these sets

1. $\{x \vert\, 0 \leq x \leq 1\}$	2. $\{x \vert\, 0 < x < 1\}$	3. $\{1 + 1/n \vert\, n \in \mathbb{N}\}$
4. $\{2 - 1/n \vert\, n \in \mathbb{N}\}$	5. $\{1 + (-1)^n/n \vert\, n \in \mathbb{N}\}$	6. $\{q \vert\, q^2 < 2,\ q \in \mathbb{Q}\}$.

For which of these sets can you find a lower bound greater than 0 and/or an upper bound less that 2? Identify the greatest lower bound and the least upper bound for each set.

Can a least upper bound or a greatest lower bound for a set A belong to the set? Must a least upper bound or a greatest lower bound for a set A belong to the set?

We have been writing *the* least upper bound so there had better be only one.

Exercise 6 Prove that a set A can have at most *one* least upper bound.

5.3 Axioms of the Real Numbers

Despite their exotic names, the following fundamental properties of the real numbers will no doubt be familiar to you. They are listed below. Just glimpse through them to check they are well known to you.

- For $x, y \in \mathbb{R}$, $x + y$ is a real number

  closure under addition

- For $x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$

  associativity of addition

- For $x, y \in \mathbb{R}$, $x + y = y + x$

  commutativity of addition

- There exists a number 0 such that for $x \in \mathbb{R}$, $x + 0 = x = 0 + x$

  existence of an additive identity

- For $x \in \mathbb{R}$ there exists a number $-x$ such that $x + (-x) = 0 = (-x) + x$

  existence of additive inverses

- For $x, y \in \mathbb{R}$, xy is a real number

  closure under multiplication

- For $x, y, z \in \mathbb{R}$, $(xy)z = x(yz)$

  associativity of multiplication

- For $x, y \in \mathbb{R}$, $xy = yx$

  commutativity of multiplication

- There exists a number 1 such that $x \cdot 1 = x = 1 \cdot x$ for all $x \in \mathbb{R}$.

  existence of multiplicative identity

- For $x \in \mathbb{R}, x \neq 0$ there exists a number x^{-1} such that $x \cdot x^{-1} = 1 = x^{-1} \cdot x$

  existence of multiplicative inverses

- For $x, y, z \in \mathbb{R}$, $x(y + z) = xy + xz$

  distributive law

- For $x, y \in \mathbb{R}$, exactly one of the following statements is true: $x < y$, $x = y$ or $x > y$

  trichotomy

- For $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$ then $x < z$

  transitivity

- For $x, y, z \in \mathbb{R}$, if $x < y$ then $x + z < y + z$

  adding to an inequality

- For $x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$ then $zx < zy$

  multiplying an inequality

There is one last axiom, without which the reals would not behave as expected:

Completeness Axiom
Every non-empty subset of the reals that is bounded above has a least upper bound.

If you lived on a planet where they only used the rational numbers then all the axioms would hold *except* the completeness axiom. The set $\{x \in \mathbb{Q} : x^2 \leq 2\}$ has rational upper bounds $1.5, 1.42, 1.415, \ldots$ but no rational least upper bound.

Of course, living in the reals we can see that the least upper bound is $\sqrt{2}$. This sort of problem arises because the rationals are riddled with holes and the completeness axiom captures our intuition that the real line has no holes in it - it is complete.

Exercise 7 If A and B denote bounded sets of real numbers, how do the numbers $\sup A$, $\inf A$, $\sup B$, $\inf B$ relate if $B \subset A$?

Give examples of unequal sets for which $\sup A = \sup B$ and $\inf A = \inf B$.

The following property of the supremum is used frequently throughout Analysis.

Possible Lack of Attainment

Notice that $\sup A$ and $\inf A$ need not be elements of A.

Lemma
Suppose a set A is non-empty and bounded above. For every $\epsilon > 0$, there exists $a \in A$ such that

$$\sup A - \epsilon < a \leq \sup A.$$

Proof. *Ab absurdo.* If the lemma is wrong, then there exists $\varepsilon > 0$ such that the interval $(\sup A - \varepsilon, \sup A]$ contains no number of A. Since A has no number greater than $\sup A$, that means that all numbers of A are less (or equal) than $\sup A - \varepsilon$. Then $\sup A - \varepsilon$ is an upper bound for A. It is smaller than $\sup A$, which contradicts the fact that $\sup A$ is the least upper bound. ■

Exercise 8 Suppose A is a non-empty set of real numbers which is bounded below. Define the set $-A = \{-a : a \in A\}$.

1. Sketch two such sets A and $-A$ on the real line. Notice that they are reflected about the origin. Mark in the position of $\inf A$.

2. Prove that $-A$ is a non-empty set of real numbers which is bounded below, and that $\sup(-A) = -\inf A$. Mark $\sup(-A)$ on the diagram.

Different Versions of Completeness

This Theorem has been named 'Greatest lower bounds *version*' because it is an equivalent version of the Axiom of Completeness. Between now and the end of the next chapter we will uncover 5 more versions!

Theorem *Greatest lower bounds version*
Every non-empty set of real numbers which is bounded below has a greatest lower bound.

Proof. Suppose A is a non-empty set of real numbers which is bounded below. Then $-A$ is a non-empty set of real numbers which is bounded above. The completeness axiom tells us that $-A$ has a least upper bound $\sup(-A)$. From Exercise 8 we know that $A = -(-A)$ has a greatest lower bound, and that $\inf A = -\sup(-A)$. ■

5.4 Consequences of Completeness - Bounded Monotonic Sequences

The mathematician Weierstrass was the first to pin down the ideas of completeness in the 1860's and to point out that all the deeper results of analysis are based upon completeness. The most immediately useful consequence is the following theorem:

Theorem *Increasing sequence version*
Every bounded increasing sequence is convergent.

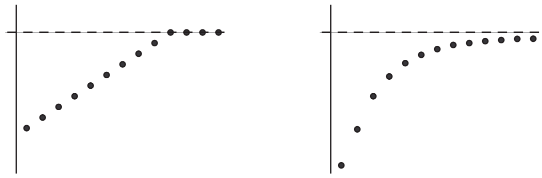

Figure 5.1: Bounded increasing sequences must converge.

Figure 5.1 should make this reasonable. Plotting the sequence on the real line as the set $A = \{a_1, a_2, a_3, \dots\}$ we can guess that the limit should be $\sup A$.
Proof. Let (a_n) be a bounded increasing sequence. We show that $a_n \to \sup A$. Let ε be any positive number. By the above lemma, there exists $a_N \in A$ such that $\sup A - \varepsilon < a_N \leq \sup A$. Since (a_n) is increasing, we have

$$\sup A - \varepsilon < a_n \leq \sup A$$

for all $n > N$. Then $|a_n - \sup A| < \varepsilon$. This holds for every $\varepsilon > 0$, so that $a_n \to \sup A$. ■

Check that your proof has used the completeness axiom, the fact that the sequence is increasing, and the fact that the sequence is bounded above. If you have not used each of these then your proof must be wrong!

Corollary *Decreasing sequence version*
Every bounded decreasing sequence is convergent.

Proof. The sequence $(-a_n)$ is bounded and increasing, then it converges to a number $-a$. Then $a_n \to a$ by the theorem of Section 2.6. ■

Example In Chapter 3, we considered a recursively defined sequence (a_n) where

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \sqrt{a_n + 2}.$$

We showed by induction that $a_n \geq 1$ for all n (because $a_1 = 1$ and $a_k \geq 1 \implies a_{k+1} = \sqrt{a_k + 2} \geq \sqrt{3} \geq 1$) and that $a_n \leq 2$ for all n (because $a_1 \leq 2$ and $a_k \leq 2 \implies a_{k+1} = \sqrt{a_k + 2} \leq \sqrt{4} = 2$). So (a_n) is bounded.

We now show that the sequence is increasing.

$$\begin{aligned} a_n^2 - a_n - 2 = (a_n - 2)(a_n + 1) \leq 0 &\text{ since } 1 \leq a_n \leq 2 \\ \therefore a_n^2 \leq a_n + 2 & \\ \therefore a_n \leq \sqrt{a_n + 2} = a_{n+1}. & \end{aligned}$$

Decreasing?

To see whether a sequence (a_n) is decreasing, try testing

$$a_{n+1} - a_n \leq 0$$

or, when terms are positive,

$$\frac{a_{n+1}}{a_n} \leq 1.$$

Hence (a_n) is increasing and bounded. It follows from Theorem 5.4 that (a_n) is convergent. Call the limit a. Then $a^2 = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} a_n + 2 = a + 2$ so that $a^2 - a - 2 = 0 \implies a = 2$ or $a = -1$. Since $(a_n) \in [1, 2]$ for all n we know from results in Chapter 3 that $a \in [1, 2]$, so the limit must be 2.

Exercise 9 Consider the sequence (a_n) defined by

$$a_1 = \frac{5}{2} \text{ and } a_{n+1} = \frac{1}{5}\left(a_n^2 + 6\right).$$

Show by induction that $2 < a_k < 3$. Show that (a_n) is decreasing. Finally, show that (a_n) is convergent and find its limit.

Exercise 10 Explain why every monotonic sequence is either bounded above or bounded below. Deduce that an increasing sequence which is bounded above is bounded, and that a decreasing sequence which is bounded below is bounded.

Exercise 11 If (a_n) is an increasing sequence that is *not* bounded above, show that $(a_n) \to \infty$. Make a rough sketch of the situation.

The two theorems on convergence of bounded increasing or decreasing sequences give us a method for showing that monotonic sequences converge even though we may not know what the limit is.

5.5 * Application - k^{th} Roots *

So far, we have taken it for granted that every positive number a has a unique positive k^{th} root, that is there exists $b > 0$ such that $b^k = a$, and we have been writing $b = a^{1/k}$. But how do we know such a root exists? We now give a careful proof. Note that even square roots do not exist if we live just with the rationals - so any proof must use the Axiom of Completeness.

Stop Press

$\sqrt{2}$ exists!!!
Mathematicians have at last confirmed that $\sqrt{2}$ is really there.

Phew! What a relief.

Theorem
Every positive real number has a unique positive k^{th} root.

Suppose a is a positive real number and k is a natural number. We wish to show that there exists a unique positive number b such that $b^k = a$. The idea of the proof is to define the set $A = \{x > 0 : x^k > a\}$ of numbers that are too big to be the k^{th} root. The infimum of this set, which we will show to exist by the greatest lower bound characterisation of completeness in this chapter, *should* be the k^{th} root. We must check this.

Note that the greatest lower bound characterisation is an immediate consequence of the completeness axiom. It is indeed equivalent to the completeness axiom, and some authors give it as the completeness axiom.

Fill in the gaps in the following proof:

Exercise 12 Show that the set A is non-empty [Hint: Show that $1+a \in A$].

By definition the set A is bounded below by 0. So the greatest lower bound characterisation of completeness implies that $b = \inf A$ must exist. Argue that for each natural number n there exists $a_n \in A$ such that $b \leq a_n < b + \frac{1}{n}$.

Exercise 13 Show that $a_n^k \to b^k$ and conclude that $b^k \geq a$.

We will now show that $b^k \leq a$, by contradiction. Assume $b^k > a$. Then $0 < \frac{a}{b^k} < 1$ so we may choose $\delta > 0$ so that $\delta < \frac{b}{k}\left(1 - \frac{a}{b^k}\right)$.

Exercise 14 Now achieve a contradiction by showing that $b - \delta \in A$. (Hint: use Bernoulli's Inequality.)

We have shown that $b^k = a$. Prove that there is no other positive k^{th} root.

Arbitrary Exponents

The existence of n^{th} roots suggests one way to define the number a^x when $a > 0$ and x is *any* real number.
If $x = m/n$ is rational and $n \geq 1$ then

$$a^x = \left(a^{1/n}\right)^m$$

If x is irrational then we know there is a sequence of rationals (x_i) which converges to x. It is possible to show that the sequence (a^{x_i}) also converges and we can try to define:

$$a^x = \lim_{i\to\infty} a^{x_i}$$

Check Your Progress
By the end of this chapter you should be able to:

- Prove that there are an infinite number of rationals and irrationals in every open interval.
- State and understand the definitions of least upper bound and greatest lower bound.
- Calculate $\sup A$ and $\inf A$ for sets on the real line.
- State and use the Completeness Axiom in the form "every non-empty set A which is bounded above has a least upper bound ($\sup A$)".