Chapter 6

Completeness II

6.1 An Interesting Sequence

Exercise 1 Consider the sequence $a_n = \left(1+\frac{1}{n}\right)^n$. Show that $\frac{a_{n+1}}{a_n} = \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^n$ and then use Bernoulli's inequality to show that $a_{n+1} \geq a_n$.

Show that $\left(1+\frac{1}{2n}\right)^n = \frac{1}{\left(1-\frac{1}{2n+1}\right)^n}$ and then use Bernoulli's inequality to show that $\left(1+\frac{1}{2n}\right)^n \leq 2$. Hence show that (a_{2n}) is bounded. Using the fact that (a_n) is increasing, show that it is bounded and hence convergent.

$e = 2.718...$

Newton showed already in 1665 that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The constant was named by Euler, who proved that

$$e = \lim_{n\to\infty} (1+\tfrac{1}{n})^n.$$

It is also the limit of $n/\sqrt[n]{n!}$ (compare with Stirling's formula!). It is known that e is irrational (Euler, 1737) and transcendental (Hermite, 1873).

Exercise 2 Show that $\left(1-\frac{1}{n}\right) = \frac{1}{\left(1+\frac{1}{n-1}\right)}$ and hence that $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$ exists.

Exercise 3 Criticise the following argument: $\left(1+\frac{1}{n}\right)^n \to (1)^n = 1$.

6.2 Consequences of Completeness - General Bounded Sequences

We showed in Chapter 3 that every subsequence of a bounded sequence is bounded. We also saw that every sequence has a monotonic subsequence (see Section 3.4). We can now tie these facts together.

Exercise 4

1. Find an upper bound and a lower bound for the sequences with n^{th} term

$$(a)\ (-1)^n, \qquad (b)\ (-1)^n\left(1+\frac{1}{n}\right).$$

   Is either sequence convergent? In each case find a convergent subsequence. Is the convergent subsequence monotonic?

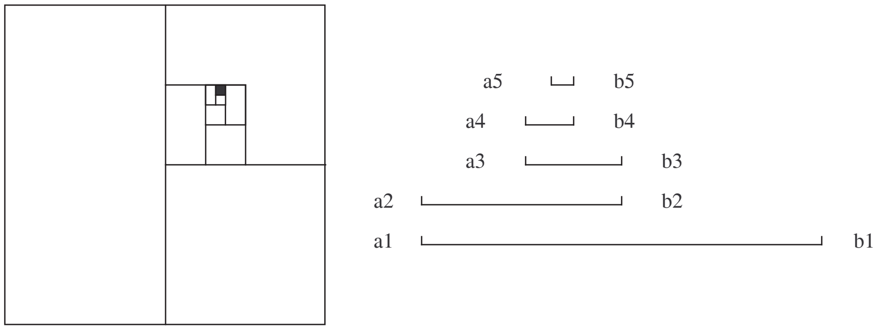

Figure 6.1: Lion Hunting.

2. Look back at your proof that every convergent sequence is bounded (Workbook 3). Is it true that every bounded sequence is convergent?

Theorem *Bolzano-Weierstrass*
Every bounded sequence has a convergent subsequence.

Example The weird, oscillating sequence $(\sin n)$ is far from being convergent. But, since $-1 \leq \sin n \leq 1$, we are guaranteed that it has a convergent subsequence.

Proof. Recall the theorem of Section 3.4: every sequence has a monotonic subsequence. If the sequence is bounded, the subsequence is also bounded, and it converges by the theorem of Section 5.4. ■

There is another method of proving the Bolzano-Weierstrass theorem called Lion Hunting - a technique useful elsewhere in analysis. The name refers to a method trapping a lion hiding in a square jungle. Build a lion proof fence dividing the jungle in half. Shortly, by listening for screams, it will be apparent in which half the lion is hiding. Build a second fence dividing this region in half. Repeating this procedure quickly traps the lion in a manageable area, see figure 6.1.

We use this idea to find a limit point for a sequence on the real line. We will illustrate this on a sequence (x_n) all of whose values lie in $[a_1, b_1] = [0, 1]$. At least one of the two intervals $[0, 1/2]$ and $[1/2, 1]$ must contain infinitely many of the points of (x_n). Choosing this half (or choosing at random if both contain infinitely many points) we label it as the interval $[a_2, b_2]$. Then we split this interval into two and we can find one of these halves which contains infinitely many of the points (x_n), and we label it $[a_3, b_3]$. We continue in this way: at the k^{th} step we start with an interval $[a_k, b_k]$ containing infinitely many points. One of the intervals $\left[a_k, \frac{a_k+b_k}{2}\right]$ or $\left[\frac{a_k+b_k}{2}, b_k\right]$ still has infinitely many points and we label this as $[a_{k+1}, b_{k+1}]$.

Exercise 5 Explain why (a_n) and (b_n) converge to a limit L. Explain why it is possible to find a subsequence (x_{n_i}) so that $x_{n_k} \in [a_k, b_k]$ and show that this subsequence is convergent.

6.3 Cauchy Sequences

Recall the notion of convergence: $a_n \to a$ if and only if for every $\varepsilon > 0$, there exists N such that $|a_n - a| < \varepsilon$ for all $n > N$. This definition has one drawback, namely that we need to know a in order to prove that the sequence converges. In Chapter 5 we found a criterion for convergence that does not involve the actual limit:

Convergence Test
A monotonic sequence converges if and only if it is bounded.

Exercise 6 Have you proved both the "if" and the "only if" parts of this test?

Is there a similar test that works for general non-monotonic sequences?

Exercise 7 Cleverclog's Test says that a sequence converges if and only if $a_{n+1} - a_n \to 0$. Give an example to show that Cleverclog's test is completely false (alas).

There is a test for convergence of a general sequence, which does not involve the limit, which we shall discover in this section.

Definition
A sequence (a_n) has the *Cauchy property* if, for each $\epsilon > 0$ there exists a natural number N such that $|a_n - a_m| < \epsilon$ for all $n, m > N$.

We use the shorthand "a Cauchy sequence" for a sequence with the Cauchy property. In words, the Cauchy property means that for any positive ϵ, no matter how small, we can find a point in the sequence beyond which any two of the terms are less than ϵ apart. So the terms are getting more and more "clustered" or "crowded".
Example $\left(\frac{1}{n}\right)$ is a Cauchy sequence. Fix $\epsilon > 0$. We have to find a natural number N such that if $n, m > N$ then

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \epsilon.$$

But

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m}.$$

Thus, if $\frac{1}{n} < \frac{\epsilon}{2}$ and $\frac{1}{m} < \frac{\epsilon}{2}$ we will have what we need. These two conditions hold when both n and m are greater than $\frac{2}{\epsilon}$. Hence we choose N to be a natural number with $N > \frac{2}{\epsilon}$. Then we have, for $n, m > N$

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

Exercise 8 Suppose $(a_n) \to a$. Show that $|a_n - a_m| \leq |a_n - a| + |a - a_m|$. Use this fact to prove that (a_n) is Cauchy.

This shows that every convergent sequence is Cauchy.

The beauty of the Cauchy property is that it is sufficient to ensure the convergence of a sequence - without having to know or show just what the limit is.

Big Bangers

Bolzano (1781-1848), Cauchy (1789-1857) and Weierstrass (1815-1897) all helped fuel the analytical Big Bang of the 19^{th} century. Both the Bolzano-Weierstrass Theorem and the theorem stating that every Cauchy sequence converges were discovered by Bolzano, a humble Czech priest. But it took Weierstrass and Cauchy to broadcast them to the world.

Theorem
Every Cauchy sequence is convergent.

Exercise 9 Let (a_n) be a Cauchy sequence. By putting $\varepsilon = 1$ in the Cauchy criterion prove that every Cauchy sequence is bounded. Now use the Bolzano-Weierstrass Theorem together with the identity

$$|a_n - a| \leq |a_n - a_{n_i}| + |a_{n_i} - a|$$

to prove that every Cauchy sequence is convergent.

Combining the last two results we have the following general test:

Convergence Test
A sequence is convergent if and only if it has the Cauchy property.

The previous theorem will be one of the most used results in your future analysis courses. Here we give only one application. A sequence (a_n) is called *strictly contracting* if for some number $0 < l < 1$, called the contraction factor,

$$|a_{n+1} - a_n| \leq l\,|a_n - a_{n-1}| \text{ for all } n = 1, 2, 3, \ldots$$

In words, the distances between successive terms are decreasing by a factor l.

Exercise 10 Define a sequence by $a_0 = 1$ and $a_{n+1} = \cos(a_n/2)$. Use the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ (which you may assume) to show that (a_n) is strictly contracting with contracting factor $l = 1/2$.

Exercise 11 The aim of this question is to show that a strictly contracting sequence (a_n) is Cauchy. Show by induction on n that $|a_{n+1} - a_n| \leq |a_1 - a_0| l^n$. Then suppose that $n > m$ and use the triangle inequality in the form:

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots |a_{m+1} - a_m|$$

to show that (a_n) is Cauchy.

If we apply this to the contracting sequence found in Exercise 10 we see that the sequence given by $a_0 = 1$ and $a_{n+1} = \cos(a_n/2)$ defines a Cauchy sequence. So by the previous theorem it must converge, say $(a_n) \to a$.

Exercise 12 Using the sequence and inequality given in Exercise 10, show that $\cos(a_n/2) \to \cos(a/2)$. Hence show that the sequence (a_n) converges to the unique solution of $x = \cos(x/2)$.

6.4 The Many Faces of Completeness

We have proved that the results below (except for the *infinite decimal sequences version* which is proved in the next section) are all consequences of the Axiom of Completeness. In fact, all of them are logically equivalent to this Axiom and to each other. This means you can prove any one of them from any other of them. So any one of them can be used as an alternative formulation of the Completeness Axiom and indeed you will find many books that use one of the results 1,2,3,4,5 or 6 as their axiom of completeness.

Completeness Axiom
Every non-empty set A of real numbers which is bounded above has a least upper bound, called $\sup A$.

Equivalent Conditions 1. 2. and 3. were proved in Chapter 5 (Completeness I).

1. Every non-empty set A of real numbers which is bounded below has a *greatest* lower bound, called $\inf A$.
2. Every bounded increasing sequence is convergent.
3. Every bounded decreasing sequence is convergent.
4. Every bounded sequence has a convergent subsequence.
5. Every Cauchy sequence is convergent.
6. Every infinite decimal sequence is convergent.

6.5 * Application - Classification of Decimals *

In this section we are going to take a close look at decimal representations for real numbers. We use expansions in base 10 (why?) but most of the results below hold for other bases: binary expansions (base 2) or ternary expansions (base 3) ...

Exercise 13

Dotty Notation
Don't forget the notation for repeating decimals:
A single dot means that that digit is repeated forever, so that $0.82\dot{3}$ stands for the infinite decimal $0.8233333\ldots$.
Two dots means that the sequence of digits between the dots is repeated forever, so $1.8\dot{2}4\dot{3}$ stands for $1.8243243243243\ldots$.

1. Write each of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, as a decimal. Which of them have finite decimal representations?

2. Find the decimal representation of $\frac{1}{17}$. Is your answer exacly $\frac{1}{17}$?

The easiest decimal representations are the finite ones - the ones that have only a finite number of decimal places, like 342.5017. A positive finite decimal has the form $d_0.d_1d_2\dots d_n$ where d_0 is a non-negative integer and each of the $d_1, d_2, \dots, d_n$ is one of the integers $0, 1, 2, \dots, 9$. Then $d_0.d_1d_2\dots d_n$ is defined to be the number:

$$d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

or, written more compactly, $\sum_{j=0}^{n} d_j 10^{-j}$.

Exercise 14 What changes are needed when defining a negative finite decimal?

The definition of an infinite decimal requires a bit more care.

Two too many

Don't get confused between the sequence (d_n) and the sequence of sums

$$\left(\sum_{j=0}^{n} d_j 10^{-j}\right)$$

The sequence (d_n) consists of the *digits* of the decimal number. The sequence of sums is the sequence of which we take the limit.

Definition
A positive real number x has a representation as an *infinite decimal* if there is a non-negative integer d_0 and a sequence (d_n) with $d_n \in \{0, 1, \dots, 9\}$ for each n, such that the sequence with n^{th}term defined by:

$$d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} = \sum_{j=0}^{n} d_j 10^{-j}$$

converges to x. In this case, we write:

$$x = d_0.d_1d_2d_3\dots$$

A negative real number x has a representation as the *infinite decimal* $(-d_0).d_1d_2d_3d_4\dots$ if $-x$ has a representation as the infinite decimal $d_0.d_1d_2d_3d_4\dots$.

Example Writing $\pi = 3.1415926\dots$ means that π is the limit of the sequence (3.1, 3.14, 3.141, 3.1415, …).

We could equally have said that the decimal expansion $d_0.d_1d_2d_3\dots$ with $d_0 \geq 0$, represents a real number x if the sequence of sums $\left(\sum_{k=0}^{n} d_k 10^{-k}\right)$ converges to x.

It is almost obvious that every real number has a decimal representation. For example, if x is positive we can find the decimal digits as follows. Define d_0

to be the largest integer less than or equal to x. Then define iteratively:

$$d_1 = \max\{j : d_0 + \frac{j}{10} \leq x\}$$
$$d_2 = \max\{j : d_0 + \frac{d_1}{10} + \frac{j}{10^2} \leq x\}$$
$$\ldots$$
$$d_n = \max\{j : d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_{n-1}}{10^{n-1}} + \frac{j}{10^n} \leq x\}$$

It is easy to check that each digit is in $\{0, 1, 2, \ldots, 9\}$. Moreover, after N digits we must have $x - \frac{1}{10^N} < \sum_{n=0}^{N} d_n 10^{-n} \leq x$ so that x is the limit of the sequence of sums by the Sandwich Rule.

Up and Down

Notice that every *non-negative* infinite decimal is the limit of *increasing* finite decimals, because you are always adding an additional non-negative term as you go.
However, every *negative* infinite decimal is the limit of *decreasing* finite decimals.

*** Consequences of Completeness for Decimals ***

At the moment, whenever we talk about a decimal expansion, $d_0.d_1d_2d_3\ldots$, we need to show that the sequence of sums converges. What would be useful is a theorem to state that this sequence *always* converges.

Theorem *Infinite decimal sequences version*
Every infinite decimal $\pm d_0.d_1d_2d_3\ldots$ represents a real number.

Exercise 15 Check that the sequence of sums is monotonic and bounded. Use the bounded increasing sequence version of the Completeness Axiom to show that the infinite decimal represents a real number.

The Decimal Dream

We know that every time we write down a list of decimal digits

$$d_0.d_1d_2d_3d_4\ldots$$

we succeed in defining a real number.

This is result 6 from earlier in the workbook. With this result, our analysis of completeness is complete.

*** Is $0.999\ldots$ Equal to 1? ***

Although we have finished our examination of completeness, there are still some things we can do with decimals.
Example What is $0.12\dot{1}\dot{2}$?

For this decimal, we have $d_0 = 0$ and the sequence (d_n) is defined by $d_{2n} = 2$ and $d_{2n+1} = 1$. Then the sequence of sums is:

$$\left(\sum_{j=0}^{n} d_j \times 10^{-j}\right)$$

We know that this converges and thus to find the limit it is sufficient to find the limit of a subsequence of the sequence of sums. The subsequence we choose is that of the even terms. This is given by:

$$\begin{aligned}
\sum_{j=0}^{2n} d_j \times 10^{-j} &= \sum_{k=1}^{n} \left(d_{2k-1}10^{-2k+1} + d_{2k}10^{-2k}\right) \\
&= \sum_{k=1}^{n} (1 \times 10 + 2) \times 10^{-2k} \\
&= \sum_{k=1}^{n} \frac{12}{100^k} \\
&= \frac{12}{100} \sum_{k=0}^{n-1} \frac{1}{100^k} \\
&= \frac{12}{100} \left(\frac{1 - \left(\frac{1}{100}\right)^n}{1 - \frac{1}{100}} \right) \\
&= \frac{12}{99} \left(1 - \left(\frac{1}{100}\right)^n \right)
\end{aligned}$$

and we can see that this converges to $\frac{12}{99}$. Thus $0.12\dot{1}\dot{2} = \frac{12}{99}$.

Exercise 16 Prove that $0.999\dot{9} = 1$.

This last exercise already shows one of the annoying features of decimals. You can have two *different* decimal representations for the *same* number. Indeed, any number with a finite decimal representation also has a representation as a decimal with recurring 9's, for example 2.15 is the same as 2.1499999....

Theorem
Suppose a positive real number has two different representations as an infinite decimal. Then one of these is finite and the other ends with a recurring string of nines.

Proof. Suppose a positive real number x has two decimal representations $a_0.a_1a_2a_3\ldots$ and $b_0.b_1b_2b_3\ldots$ and that the decimal places agree until the N^{th} place where $a_N < b_N$. Then:

$$
\begin{aligned}
x &= \lim_{n\to\infty} \sum_{k=0}^{n} a_k 10^{-k} \\
&= \sum_{k=0}^{N} a_k 10^{-k} + \lim_{n\to\infty} \sum_{k=N+1}^{n} a_k 10^{-k} \\
&\leq \sum_{k=0}^{N} a_k 10^{-k} + \lim_{n\to\infty} \sum_{k=N+1}^{n} 9 \times 10^{-k} \\
&= \sum_{k=0}^{N} a_k 10^{-k} + \frac{9}{10^{N+1}} \lim_{n\to\infty} \frac{1-\left(\frac{1}{10^{n-N}}\right)}{1-\frac{1}{10}} \\
&= \sum_{k=0}^{N} a_k 10^{-k} + \frac{1}{10^N} \\
&= \sum_{k=0}^{N-1} a_k 10^{-k} + (a_N + 1)10^{-N} \\
&\leq \sum_{k=0}^{N} b_k 10^{-k} && \text{as } a_N < b_N \\
&\leq \lim_{n\to\infty} \sum_{k=0}^{n} b_k 10^{-k} && \text{as the sequence is increasing} \\
&= x
\end{aligned}
$$

■

Since we started and ended with the number x, the above inequalities must all be equalities. So it is possible to have two decimal representations for x provided all the decimal digits a_n are 9 when $n > N$ and all the digits b_n are 0 when $n > N$ and $b_N = a_N + 1$. If even a single one of these digits fails to be a 9 (respectively a 0) then the above chain of inequalities becomes a strict inequality and we reach the contradiction $x < x$.

Recurring Nines

The problem of non-uniqueness of decimal representations is annoying but not too bad. In many problems we can just agree to use one of the two representations - for instance by banning any representation that has recurring nines.

* Classifying Decimals *

We now classify decimals into three types.

Definition
An infinite decimal $\pm d_0.d_1d_2d_3d_4\ldots$ is

terminating	if it ends in repeated zeros i.e. there exists N such that $d_n = 0$ whenever $n > N$.
recurring	if it eventually repeats itself i.e. there exist N and r such that $d_n = d_{n+r}$ whenever $n > N$.
non-recurring	if it is neither terminating nor recurring

Examples

- 532.89764 is terminating.
- $0.333\dot{3}$ is recurring.
- $3.1415\ldots$, the decimal expansion of π, is nonrecurring.

You can see that a terminating decimal is really just a finite decimal in disguise. It is also an example of a recurring decimal, since it ends with a string of repeated zeros.

*** Terminating Decimals ***

Exercise 17 Suppose $x = p/q$ for integers p, q where the only prime factors of q are 2's and 5's. Show that x has a terminating decimal representation. [Hint: show that $x = p'/10^n$ for some integer p' and some $n \geq 0$.]

Exercise 18 Show that if x has a terminating decimal expansion then $x = p/q$ for integers p, q where the only prime factors of q are $2's$ and $5's$.

Together the last two exercises have shown the following theorem:

Theorem *Characterisation of terminating decimals*
A number x can be represented by a terminating decimal if and only if $x = p/q$ for integers p, q where the only prime factors of q are 2's and 5's.

*** Recurring Decimals ***

Exercise 19 Express the recurring decimal $1.23\dot{4}5\dot{6}$ as a fraction.

Working through this example should convince you of the following.

Theorem
Every recurring decimal represents a rational number.

Exercise 20 To show this, suppose that x has a decimal representation that has recurring blocks of length k. Explain why $10^k x - x$ must have a terminating decimal representation. Now use the characterisation of terminating decimals to show that $x = \frac{p}{q(10^k - 1)}$ for some integers p, q where q has no prime factors except 2's and 5's.

Corollary
A recurring decimal x with repeating blocks of length k can be written as $x = \frac{p}{q(10^k - 1)}$ where the only prime factors of q are 2's or 5's.

Exercise 21

1. Express $\frac{333}{22}$ as a recurring decimal.

2. Use long division to express $\frac{1}{7}$ as a recurring decimal. Write out the long division sum explicitly (don't use a calculator). In your long division circle the remainders after each subtraction step. Are all the possible remainders 0,1,2,3,4,5,6 involved? How long is the repeating block?

3. Use long division to express $\frac{1}{13}$ as a recurring decimal. In your long division circle the remainders after each subtraction step. Are all the possible remainders $0, 1, 2, \ldots, 11, 12$ involved? How long is the repeating block?

The exercise above should convince you of the following result:

Theorem
Every rational number can be represented by a *recurring* infinite decimal or a terminating infinite decimal.

*** Complete Classification ***

We now have a complete understanding of recurring decimals. Recurring decimals represent rationals and rationals always have recurring decimal representations. What about non-recurring decimals? Since every number has a decimal representation, it follows that any irrational number must have a non-recurring infinite decimal representation.

Theorem
Every real number has a decimal representation and every decimal represents a real number.
The *rationals* are the set of terminating or recurring decimals.
The *irrationals* are the set of non-recurring decimals.
If a number has two distinct representations then one will terminate and the other will end with a recurring string of nines.

The decimal system

The earliest evidence of the decimal system goes back to China, in the 13th century B.C. The zero makes a furtive appearance in the 4th century B.C. in China where it is represented by a space, and in the 2nd century B.C. in Babylon where it is represented by two small wedges. At about the same time, a proper symbol appears in India that is called "sunya" in Sanskrit. This word will travel through the Perso-Arabic World as "sifr", which will give both the words "zero" and "cipher". Fibonacci (1170–1250) introduced the decimal system in Europe.
Notice that the Greek and Roman civilisations did not know about the zero. The Babylonians did not use the decimal system, choosing 60 instead; neither did Precolombians civilisations like the Mayas, choosing 20.

Check Your Progress
By the end of this chapter you should be able to:

- Prove and use the fact that every bounded increasing sequence is convergent.
- Prove and use the fact that every bounded decreasing sequence is convergent.
- Prove the Bolzano-Weierstrass Theorem: that every bounded sequence of real numbers has a convergent subsequence.
- State, use, and understand the definition of a Cauchy sequence.
- Prove that a sequence of real numbers is convergent if and only if it is Cauchy.
- Understand and be able to use the definition of a decimal expansion.