Chapter 7

Series I

7.1 Definitions

We saw in the last booklet that decimal expansions could be defined in terms of sequences of sums. Thus a decimal expansion is like an infinite sum. This is what we shall be looking at for the rest of the course.

Our first aim is to find a good definition for summing infinitely many numbers. Then we will investigate whether the rules for finite sums apply to infinite sums.

Exercise 1 What has gone wrong with the following argument? Try putting $x = 2$.

> If $S = 1 + x + x^2 + \dots,$
> then $xS = x + x^2 + x^3 + \dots,$
> so $S - xS = 1,$
> and therefore $S = \frac{1}{1-x}$.

If the argument were correct then we could put $x = -1$ to obtain the sum of the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots$$

as 1/2. But the same series could also be be thought of as

$$(1 - 1) + (1 - 1) + (1 - 1) + \dots$$

with a sum of 0, or as

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots$$

with a sum of 1. This shows us that great care must be exercised when dealing with infinite sums.

We shall repeatedly use the following convenient notation for finite sums: given integers $0 \le m \le n$ and numbers $(a_n : n = 0, 1, \dots)$ we define

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \dots + a_n$$

Example $1 + 4 + 9 + \cdots + 100 = \sum_{k=1}^{10} k^2$

Exercise 2 Express the following sums using the $\sum$ notation:

$$1.\ \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{3628800} \qquad 2.\ \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots \frac{7}{128}$$

Exercise 3 Show that $\sum_{k=1}^{n} a_{k-1} = \sum_{k=0}^{n-1} a_k$.

Exercise 4

1. By decomposing $1/r(r+1)$ into partial fractions, or by induction, prove that
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$
Write this result using $\sum$ notation.

2. If
$$s_n = \sum_{r=1}^{n} \frac{1}{r(r+1)},$$
prove that $(s_n) \to 1$ as $n \to \infty$. This result could also be written as
$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)} = 1.$$

Serious Sums

The problem of how to deal with infinite sums vexed the analysis of the early 19th century. Some said there wasn't a problem, some pretended there wasn't until inconsistencies in their own work began to unnerve them, and some said there was a terrible problem and why wouldn't anyone listen? Eventually, everyone did.

A *series* is an expression of the form $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$. As yet, we have not defined what we mean by such an infinite sum. To get the ball rolling, we consider the "partial sums" of the series:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

To have any hope of computing the infinite sum $a_1 + a_2 + a_3 + \ldots$, then the partial sums s_n should represent closer and closer approximations as n is chosen larger and larger. This is just an informal way of saying that the infinite sum $a_1 + a_2 + a_3 + \ldots$ ought to be the limit of the sequence of partial sums.

Definition
Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ with partial sums (s_n), where

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

We say:

1. $\sum_{n=1}^{\infty} a_n$ *converges* if (s_n) converges. If $s_n \to S$ then we call S the sum of the series and we write $\sum_{n=1}^{\infty} a_n = S$.
2. $\sum_{n=1}^{\infty} a_n$ *diverges* if (s_n) does not converge.
3. $\sum_{n=1}^{\infty} a_n$ *diverges to infinity* if (s_n) tends to infinity.
4. $\sum_{n=1}^{\infty} a_n$ *diverges to minus infinity* if (s_n) tends to minus infinity.

We sometimes write the series $\sum_{n=1}^{\infty} a_n$ simply as $\sum a_n$.
Example Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$. The sequence of partial sums is given by

$$s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{1}{2}\left(\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\right) = 1 - \left(\frac{1}{2}\right)^n$$

Clearly $s_n \to 1$. It follows from the definition that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

We could express the argument more succinctly by writing

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} \left(1 - \left(\frac{1}{2}\right)^n\right) = 1$$

Example Consider the series $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + \dots$. Here we have the partial sums:

$$\begin{aligned} s_1 &= a_1 = -1 \\ s_2 &= a_1 + a_2 = 0 \\ s_3 &= a_1 + a_2 + a_3 = -1 \\ s_4 &= a_1 + a_2 + a_3 + a_4 = 0 \\ &\dots\dots\dots \end{aligned}$$

and we can see at once that the sequence $(s_n) = -1, 0, -1, 0, \dots$ does not converge.

Exercise 5 Look again at the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Plot on two small separate graphs both the sequences $(a_n) = \left(\frac{1}{2^n}\right)$ and $(s_n) = \left(\sum_{k=1}^{n} \frac{1}{2^k}\right)$.

Double Trouble
There are two sequences associated with every series $\sum_{n=1}^{\infty} a_n$: the sequence (a_n) and the sequence of partial sums $(s_n) = \left(\sum_{i=1}^{n} a_i\right)$. Do not get these sequences confused!

Series Need Sequences
Notice that series convergence is defined entirely in terms of sequence convergence. We haven't spent six weeks working on sequences for nothing!

Dummy Variables
Make careful note of the way the variables k and n appear in this example. They are dummies - they can be replaced by any letter you like.

Frog Hopping
Heard about that frog who hops halfway across his pond, and then half the rest of the way, and the half that, and half that, and half that ...?
Is he ever going to make it to the other side?

Exercise 6 Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{1}{10^n}\right)$.

Exercise 7 Reread your answer to exercise 4 and then write out a full proof that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots=1$$

Exercise 8 Show that the series $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{5}+\dots$ diverges to $+\infty$. [Hint: Calculate the partial sums $s_1, s_3, s_6, s_{10}, \dots$.]

7.2 Geometric Series

Theorem *Geometric Series*
The series $\sum_{n=0}^{\infty} x^n$ is convergent if $|x|<1$ and the sum is $\frac{1}{1-x}$. It is divergent if $|x|\geq 1$.

Exercise 9 $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\dots$ converges.
$\sum_{n=0}^{\infty}(2.1)^n$, $\sum_{n=0}^{\infty}(-1)^n$ and $\sum_{n=0}^{\infty}(-3)^n = 1-3+9-27+81-\dots$ all diverge.

GP Consultation

How could you ever forget that $a+ax+ax^2+\cdots+ax^{n-1}=a\left(\frac{x^n-1}{x-1}\right)$ when $x\neq 1$?

Exercise 10 Prove the theorem [Hint: Use the GP formula to get a formula for s_n].

7.3 The Harmonic Series

The series $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ is called the Harmonic Series. The following grouping of its terms is rather cunning:

$$1+\underbrace{\frac{1}{2}}_{\geq 1/2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geq 1/2}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq 1/2}+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{\geq 1/2}+\dots$$

Harmonic History

There are other proofs that the Harmonic Series is divergent, but this is the original. It was contributed by the English mediaeval mathematician Nicholas Oresme (1323-1382) who also gave us the laws of exponents: $x^m.x^n=x^{m+n}$ and $(x^m)^n=x^{mn}$.

Exercise 11 Prove that the Harmonic Series diverges. Structure your proof as follows:

1. Let $s_n=\sum_{k=1}^{n}\frac{1}{k}$ be the partial sum. Show that $s_{2n}\geq s_n+\frac{1}{2}$ for all n. (Use the idea in the cunning grouping above).
2. Show by induction that $s_{2^n}\geq 1+\frac{n}{2}$ for all n.
3. Conclude that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Conflicting Convergence

You can see from this example that the convergence of (a_n) does not imply the convergence of $\sum_{n=1}^{\infty}a_n$.

Exercise 12 Give, with reasons, a value of N for which $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N} \geq 10$.

7.4 Basic Properties of Convergent Series

Some properties of finite sums are easy to prove for infinite sums:

Theorem *Sum Rule for Series*
Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series. Then, for all real numbers c and d, $\sum_{n=1}^{\infty}(ca_n + db_n)$ is a convergent series and

$$\sum_{n=1}^{\infty}(ca_n + db_n) = c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n$$

Proof.

$$\begin{aligned} \sum_{i=1}^{n}(ca_n + db_n) &= c\left(\sum_{i=1}^{n} a_i\right) + d\left(\sum_{i=1}^{n} b_i\right) \\ &\to c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n \end{aligned}$$

■

Theorem *Shift Rule for Series*
Let N be a natural number. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_{N+n}$ converges.

Example We showed that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. It follows that $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent.

Exercise 13 Prove the shift rule.

7.5 Boundedness Condition

If the terms of a series are all non-negative, then we shall show that the boundedness of its partial sums is enough to ensure convergence.

Theorem *Boundedness Condition*
Suppose $a_n \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $(s_n) = \left(\sum_{j=1}^{n} a_j\right)$ is bounded.

Proof. The sequence $(\sum_{k=1}^{n} a_k)$ is increasing. We saw in Section 6.3 that the sequence either converges, or it diverges to infinity. If it is bounded, it must converge. ■

7.6 Null Sequence Test

Exercise 14

1. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then the sequence (a_n) tends to zero. (Hint: Notice that $a_{n+1} = s_{n+1} - s_n$ and use the Shift Rule for sequences.)
2. Is the converse true: If $(a_n) \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges?

We have proved that if the series $\sum_{n=1}^{\infty} a_n$ converges then it must be the case that (a_n) tends to zero. The contrapositive of this statement gives us a test for *divergence*:

Red Alert
The Null Sequence Test is a test for *divergence* only. You can't use it to prove series convergence.

Theorem *Null Sequence Test*
If (a_n) does not tend to zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example The sequence (n^2) does not converge to zero, therefore the series $\sum_{n=1}^{\infty} n^2$ diverges.

7.7 Comparison Test

The next test allows you to test the convergence of a series by comparing its terms with those of a series whose behaviour you already know.

Theorem *Comparison Test*
Suppose $0 \leq a_n \leq b_n$ for all natural numbers n. If $\sum b_n$ converges then $\sum a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Like the Boundedness Condition, you can only apply the Comparison Test (and the other tests in this section) if the terms of the series are non-negative.

Example You showed in assignment 7 that $\sum \frac{1}{n(n+1)}$ converges. Now $0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$. It follows from the Comparison Test that $\sum \frac{1}{(n+1)^2}$ also converges and via the Shift Rule that the series $\sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ converges.

Exercise 15 Give an example to show that the test fails if we allow the terms of the series to be negative, i.e. if we only demand that $a_n \leq b_n$.

Exercise 16 Prove the Comparison Test [Hint: Consider the partial sums of both $\sum b_n$ and $\sum a_n$ and show that the latter is increasing and bounded].

Exercise 17 Check that the *contrapositive* of the statement: *"If $\sum b_n$ converges then $\sum a_n$ converges."* gives you the following additional comparison test:

Corollary *Comparison Test*
Suppose $0 \le a_n \le b_n$. If $\sum a_n$ diverges then $\sum b_n$ diverges.

Examples

1. Note $0 \le \frac{1}{n} \le \frac{1}{\sqrt{n}}$. We know $\sum \frac{1}{n}$ diverges, so $\sum \frac{1}{\sqrt{n}}$ diverges too.
2. To show that $\sum \frac{n+1}{n^2+1}$ diverges, notice that $\frac{n+1}{n^2+1} \ge \frac{n}{n^2+n^2} = \frac{1}{2n}$. We know that $\sum \frac{1}{2n}$ diverges, therefore $\sum \frac{n+1}{n^2+1}$ diverges.

Exercise 18 Use the Comparison Test to determine whether each of the following series converges or diverges. In each case you will have to think of a suitable series with which to compare it.

$$\text{(i)} \quad \sum \frac{2n^2+15n}{n^3+7} \qquad \text{(ii)} \quad \sum \frac{\sin^2 nx}{n^2} \qquad \text{(iii)} \quad \sum \frac{3^n+7^n}{3^n+8^n}$$

Way To Go
Stare deeply at each series and try to find a simpler series whose terms are very close for large n. This gives you a good idea which series you might hope to compare it with, and whether it is likely to be convergent or divergent. For instance the terms of the series $\sum \frac{n+1}{n^2+1}$ are like those of the series $\sum \frac{1}{n}$ for large values of n, so we would expect it to diverge.

7.8 * Application - What is e? *

Over the years you have no doubt formed a working relationship with the number e, and you can say with confidence (and the aid of your calculator) that $e \approx 2.718$. But that is not the end of the story.

Just what is this e number?

To answer this question, we start by investigating the mysterious series $\sum_{n=0}^{\infty} \frac{1}{n!}$. Note that we adopt the convention that $0! = 1$.

Exercise 19 Consider the series $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ and its partial sums $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!}$.

1. Show that the sequence (s_n) is increasing.
2. Prove by induction that $\frac{1}{n!} \le \frac{1}{2^{n-1}}$ for $n > 0$.
3. Use the comparison test to conclude that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Now here comes the Big Definition we've all been waiting for...!!!

Definition
$e := \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

Binomial Theorem
For all real values x and y and integer $n = 1, 2, \ldots$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Note here we use $0! = 1$.

Recall that in the last chapter we showed that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists. We can now show, with some rather delicate work, that this limit equals e.

First, we show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \le e$. Using the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{1}{k!} \underbrace{\frac{n(n-1)\ldots(n-k+1)}{n \cdot n \cdot \ldots \cdot n}}_{\le 1} \le \sum_{k=0}^{n} \frac{1}{k!}.$$

As $n \to \infty$, we get $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \le \mathrm{e}$.

Second, we show that $\mathrm{e} \le \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$. This is more difficult! The first step is to show that for all m and n

$$\left(1+\frac{1}{n}\right)^{m+n} \ge \sum_{k=0}^{m} \frac{1}{k!}$$

By the Binomial theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{m+n} &= \sum_{k=0}^{n+m} \frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k} \\ &\ge \sum_{k=0}^{m} \frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k}\end{aligned}$$

where we have thrown away the last n terms of the sum. So

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{m+n} &\ge \sum_{k=0}^{m} \frac{1}{k!}\frac{(n+m)(n+m-1)\dots(n+m-k+1)}{n^k} \\ &\ge \sum_{k=0}^{m} \frac{1}{k!}.\end{aligned}$$

We have for all $m, n \ge 1$:

$$\left(1+\frac{1}{n}\right)^{m}\left(1+\frac{1}{n}\right)^{n} \ge \sum_{k=0}^{m} \frac{1}{k!}.$$

Taking the limit $n \to \infty$, we get

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \ge \sum_{k=0}^{m} \frac{1}{k!}.$$

Taking now the limit $m \to \infty$, we get $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \ge e$.

We have proved:

Theorem
$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

Exercise 20

1. Show that $\left(1-\frac{1}{n+1}\right) = \frac{1}{(1+1/n)}$ and hence find $\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n$.
2. Use the shift rule to find $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$.

The last exercise in this chapter is the proof that e is an irrational number. The proof uses the fact that the series for e converges very rapidly and this same idea can be used to show that many other series also converge to irrational numbers.

Theorem
e is irrational.

Exercise 21 (Not easy) Prove this result by contradiction. Structure your proof as follows:

1. Suppose $e = \frac{p}{q}$ and show that $e - \sum_{i=0}^{q} \frac{1}{i!} = \frac{p}{q} - \sum_{i=0}^{q} \frac{1}{i!} = \frac{k}{q!}$ for some positive integer k.
2. Show that $e - \sum_{i=0}^{q} \frac{1}{i!} = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \dots < \frac{1}{q!}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$ and derive a contradiction to part 1.

Check Your Progress
By the end of this chapter you should be able to:

- Understand that a series converges if and only if its partial sums converge, in which case $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i\right)$.
- Write down a list of examples of convergent and divergent series and justify your choice.
- Prove that the *Harmonic Series* is divergent.
- State, prove, and use the *Sum* and *Shift Rules* for series.
- State, prove, and use the *Boundedness Condition*.
- Use and justify the *Null Sequence Test*.
- Describe the behaviour of the *Geometric Series* $\sum_{n=1}^{\infty} x^n$.
- State, prove and use the *Comparison Theorem* for series.
- Justify the limit $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ starting from the definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.
- Prove that e is irrational.