Chapter 8

Series II

Usually we are doomed to failure if we seek a formula for the sum of a series. Nevertheless we can often tell whether the series converges or diverges without explicitly finding the sum. To do this we shall establish a variety of convergence tests that allow us in many cases to work out from the formula for the terms a_n whether the series converges or not.

Explicit Sums

For most convergent series there is no simple formula for the sum $\sum_{n=1}^{\infty} a_n$ in terms of standard mathematical objects. Only in very lucky cases can we sum the series explicitly, for instance geometric series, telescoping series, various series found by contour integration or by Fourier expansions. But these cases are so useful and so much fun that we mention them often.

8.1 Series with positive terms

Series with positive terms are easier than general series since the partial sums (s_n) form an increasing sequence and we have already seen that monotonic sequences are easier to cope with than general sequences.

All our convergence tests are based on the most useful test - the comparison test - which you have already proven.

Sometimes the series of which we want to find the sum looks quite complicated. Often the best way to find a series to compare it with is to look at which terms dominate in the original series.

Example Consider the series $\sum \frac{\sqrt{n}+2}{n^{3/2}+1}$. We can rearrange the n^{th} term in this series as follows:

$$\frac{\sqrt{n}+2}{n^{3/2}+1} = \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}}.$$

As n gets large then $\frac{1}{\sqrt{n}}$ gets small so the dominant term in the numerator is the 1 and in the denominator is the n. Thus a possible series to compare it with is $\sum \frac{1}{n}$. Since this diverges, we want to show that our series is greater than some multiple of $\sum \frac{1}{n}$:

$$\begin{aligned}\frac{\sqrt{n}+2}{n^{3/2}+1} &= \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}} \\ &> \frac{1+\frac{2}{\sqrt{n}}}{2n} \\ &> \frac{1}{2n}.\end{aligned}$$

hence by the comparision test, $\sum \frac{\sqrt{n}+2}{n^{3/2}+1}$ diverges.

Look Where You're Going!

As with many of the results in this course, the Comparison Test requires you to know in advance whether you are trying to prove convergence or divergence. Otherwise you may end up with a comparison that is no use!

Exercise 1 Use the Comparison Test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p \in [2, \infty)$ and diverges if $p \in (0, 1]$. [Hint - you already know the answer for $p = 1$ or 2.]

Exercise 2 Use this technique with the Comparison Test to determine whether each of the following series converges or diverges. Make your reasoning clear.

$$1. \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \qquad 2. \sum_{n=1}^{\infty} \frac{5^n + 4^n}{7^n - 2^n}$$

Exercise 3 Use the Comparison Test to determine whether each of the following series converges or diverges.

$$1. \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}} \qquad 2. \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^7+1}} \qquad 3. \sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$$

8.2 Ratio Test

The previous tests operate by comparing two series. Choosing a Geometric Series for such a comparison gives rise to yet another test which is simple and easy but unsophisticated.

Theorem *Ratio Test*
Suppose $a_n > 0$ for all $n \geq 1$ and $\frac{a_{n+1}}{a_n} \to \ell$. Then $\sum a_n$ converges if $0 \leq \ell < 1$ and diverges if $\ell > 1$.

The Missing Case
The case $\ell = 1$ is omitted from the statement of the Ratio Test. This is because there exist both convergent *and* divergent series that satisfy this condition.

A.K.A.
This test is also called D'Alembert's Ratio Test, after the French mathematician Jean Le Rond D'Alembert (1717 - 1783). He developed it in a 1768 publication in which he established the convergence of the Binomial Series by comparing it with the Geometric Series.

Examples

1. Consider the series $\sum \frac{1}{n!}$. Letting $a_n = \frac{1}{n!}$ we obtain $\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$. Therefore $\sum \frac{1}{n!}$ converges.

2. Consider the series $\sum \frac{n^2}{2^n}$. Letting $a_n = \frac{n^2}{2^n}$ we obtain $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2}$. Therefore $\sum \frac{n^2}{2^n}$ converges.

Proof.
(a) Convergence if $\ell < 1$. Since $\frac{a_{n+1}}{a_n} \to \ell < 1$, there exists N such that $\frac{a_{n+1}}{a_n} < \frac{\ell+1}{2} < 1$ for all $n > N$. (There is nothing special about the number $\frac{\ell+1}{2}$, we only need a number that is strictly greater than ℓ and strictly less than 1.) Then

$$a_{n+1} < \frac{\ell+1}{2} a_n < \left(\frac{\ell+1}{2}\right)^2 a_{n-1} < \ldots < \left(\frac{\ell+1}{2}\right)^{n-N+1} a_N.$$

Figure 8.1: Calculating a lower bound of an integral.

By the comparison test,

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} \left(\frac{\ell+1}{2}\right)^{n-N+1} a_N$$
$$= \sum_{n=1}^{N} a_n + \left(\frac{\ell+1}{2}\right)^{-N+1} a_N \sum_{n=N+1}^{\infty} \left(\frac{\ell+1}{2}\right)^{n} < \infty.$$

The expression in the last line is finite indeed, because the first sum involves finitely many terms, and the second sum is a geometric series with number less than 1. Then $\sum a_n$ converges.

(b) Divergence if $\ell > 1$. This is similar as above — a bit simpler, actually. Since $\frac{a_{n+1}}{a_n} \to \ell > 1$, there exists N such that $\frac{a_{n+1}}{a_n} \geq 1$ for all $n > N$. (It is important here that ℓ be strictly greater than 1, this would not be true in general if $\ell = 1$.) Then

$$a_{n+1} \geq a_n \geq a_{n-1} \geq \ldots \geq a_N.$$

By the comparison test,

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=N}^{\infty} \geq a_N \sum_{n=N}^{\infty} 1 = \infty.$$

Then $\sum a_n$ diverges to infinity. ■

Exercise 4 Write down an example of a convergent series and a divergent series both of which satisfy the condition $\ell = 1$. [This shows why the Ratio Test cannot be used in this case.]

Exercise 5 Use the Ratio Test to determine whether each of the following series converges or diverges. Make your reasoning clear.

$$1. \sum \frac{2^n}{n!} \qquad 2. \sum \frac{3^n}{n} \qquad 3. \sum \frac{n!}{n^n}$$

8.3 Integral Test

We can use our integration skills to get hugely useful approximations to sums. Consider a real-valued function $f(x)$ which is non-negative and decreasing for $x \geq 1$. We have sketched such a function in Figure 8.1 (actually we sketched $f(x) = 1/x$).

Forward and Back

In later Analysis courses you will formally define both the integral and the logarithm function. Using what you know from A-level for the moment gives us access to more interesting examples.

The shaded blocks lie under the graph of the function so that the total area of all the blocks is less than the area under the graph between $x = 1$ and $x = n$. So:

$$\sum_{k=2}^{n} f(k) \leq \int_{1}^{n} f(x)dx$$

The same argument gives more general upper bounds. It also gives lower bounds, when the blocks are chosen so that their area *contains* the area below the curve. Precisely, one can prove the following claims.

Theorem *Integral bounds*
Suppose that $f(x)$ is a non-negative and decreasing function. Then for all $m \leq n$,

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx.$$

We can use this bound to help us with error estimates. Let us consider our favorite series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which turns out to be equal to $\frac{\pi^2}{6}$. If we sum only the first N terms of this series we will reach a total less than $\frac{\pi^2}{6}$. Can we estimate the size of the error?

The error is precisely $\sum_{k=N+1}^{\infty} \frac{1}{k^2}$. Using the theorem above, we obtain the bound:

$$\sum_{k=N+1}^{n} \frac{1}{k^2} \leq \int_{N}^{n} \frac{1}{x^2}dx = -\frac{1}{x}\Big|_{N}^{n} = \frac{1}{N} - \frac{1}{n} \leq \frac{1}{N}$$

Since this is true for any value of n we see that $\sum_{k=N+1}^{\infty} \frac{1}{k^2} = \lim_{n\to\infty} \sum_{k=N+1}^{n} \frac{1}{k^2} \leq \frac{1}{N}$.

So if we sum the first 1,000,000 terms we will reach a total that is within 10^{-6} of $\pi^2/6$.

Exercise 6 Fourier analysis methods also lead to the formula:

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots$$

Find a value of N so that $\sum_{k=1}^{N} \frac{1}{k^4}$ is within 10^{-6} of $\pi^4/90$.

Exercise 7 Use the upper and lower bounds in the theorem above to show

$$\sum_{k=101}^{200} \frac{1}{k} = \frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{200} \in [0.688, 0.694]$$

We now use these upper and lower bounds to establish a beautiful test for convergence.

Corollary *Integral Test*
Suppose that the function $f(x)$ is non-negative and decreasing for $x \geq 1$.

(a) If $\lim_{N\to\infty} \int_1^N f(x)dx < \infty$, then $\sum_{n=1}^{\infty} f(n)$ converges.

(b) If $\lim_{N\to\infty} \int_1^N f(x)dx = \infty$, then $\sum_{n=1}^{\infty} f(n)$ diverges.

Example The Integral Test gives us another proof of the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Let $f(x) = \frac{1}{x^2}$. We know that this function is non-negative and decreasing when $x \geq 1$. Observe that $\int_1^n f(x)dx = \int_1^n \frac{1}{x^2} = -\frac{1}{x}\Big|_1^n = 1 - \frac{1}{n} \to 1$. Since $f(n) = \frac{1}{n^2}$, the Integral Test assures us that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example If you are familiar with the behaviour of the log function, the Integral Test gives you a neat proof that the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Suppose $f(x) = \frac{1}{x}$. Again, this function is non-negative and decreasing when $x \geq 1$. Observe that $\int_1^n f(x)dx = \int_1^n \frac{1}{x}dx = \log x\Big|_1^n = \log n \to \infty$. Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity.

Exercise 8 Use the Integral Test to investigate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for values of $p \in (1, 2)$.

Combining this with the result of exercise 1, you have shown:

Corollary
The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

We now examine some series right on the borderline of convergence.

Exercise 9 Show that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ is divergent and that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent.

8.4 * Application - Error Bounds *

If we have established that a series $\sum a_n$ converges then the next question is to calculate the total sum $\sum_{n=1}^{\infty} a_n$. Usually there is no explicit formula for the sum and we must be content with an approximate answer - for example, correct to 10 decimal places.

The obvious solution is to calculate $\sum_{n=1}^{N} a_n$ for a really large N. But how large must N be to ensure the error is small - say less than 10^{-10}? The error is the sum of all the terms we have ignored $\sum_{n=N+1}^{\infty} a_n$ and again there is usually no explicit answer. But by a comparison with a series for which we *can* calculate the sum (i.e. geometric or telescoping series) we can get a useful upper bound on the error.

Example Show how to calculate the value of e to within an error of 10^{-100}.

Solution We shall sum the series $\sum_{n=0}^{N} \frac{1}{n!}$ for a large value of N. Then the error is:

$$\begin{aligned} e - \sum_{n=0}^{N} \frac{1}{n!} &= \sum_{n=N+1}^{\infty} \frac{1}{n!} \\ &= \frac{1}{(N+1)!}\left(1 + \frac{1}{N+2} + \frac{1}{(N+2)(N+3)} \cdots\right) \\ &\leq \frac{1}{(N+1)!}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \\ &= \frac{2}{(N+1)!} \end{aligned}$$

Then the error is less than 10^{-100} provided that $\frac{2}{(N+1)!} \leq 10^{-100}$ which occurs when $N \geq 70$.

Exercise 10 The following formula for $\sqrt{e}$ is true, although it will not be proved in this course.

$$\sqrt{e} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \dots$$

Show that the error $\sqrt{e} - \sum_{n=0}^{N} \frac{1}{2^n n!}$ is less than $\frac{1}{2^N (N+1)!}$. Hence find a value of N that makes the error less than 10^{-100}.

8.5 * Euler's product formula *

In this section we discuss the facinating formula of Euler that involves a product over all prime numbers, and its relation with Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Theorem *Euler's product formula*
For all $s > 1$, we have

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)},$$

where the product is over all prime numbers $p = 2, 3, 5, 7, ...$

Since $(1 - \frac{1}{p^s})^{-1} = \frac{1}{1-p^{-s}}$, the formula can also be written

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \,\text{prime}} \frac{1}{1 - p^{-s}}.$$

Proof. We establish the latter formula. Recall the formula for geometric series

$\frac{1}{1-a} = 1 + a + a^2 + ...$ We have

$$\begin{aligned}
\prod_{p\,\text{prime}} \frac{1}{1-p^{-s}} &= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot ... \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \dots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \dots\right) ... \\
&= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s \cdot 3^s} + ... \\
&= \zeta(s).
\end{aligned}$$

Indeed, all combinations of products of prime numbers appear when we expand the product of sums. We then obtain all natural numbers, since each natural number has a unique decomposition into prime numbers. ■

Check Your Progress

By the end of this chapter you should be able to:

- Use and justify the following tests for sequence convergence:
- *Comparison Test:* If $0 \leq a_n \leq b_n$ and $\sum b_n$ is convergent then $\sum a_n$ is convergent.
- *Ratio Test:* If $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$ then $\sum a_n$ converges if $0 \leq \ell < 1$ and diverges if $\ell > 1$.
- *Integral Test:* If $f(x)$ is non-negative and decreasing for $x \geq 1$ then $\sum f(n)$ converges if and only if $\int_1^\infty f(x)dx < \infty$, and $\sum f(n)$ diverges to infinity if and only if $\int_1^\infty f(x)dx = \infty$.
- You should also be able to use comparisons to establish error bounds when evalutating infinite sums.