Chapter 9

Series III

With the exception of the Null Sequence Test, all the tests for series convergence and divergence that we have considered so far have dealt only with series of non-negative terms. Series with *both* positive and negative terms are harder to deal with.

9.1 Alternating Series

One very special case is a series whose terms alternate in sign from positive to negative. That is, series of the form $\sum(-1)^{n+1}a_n$ where $a_n \geq 0$.
Example $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ is an alternating series.

Theorem *Alternating Series Test*
Suppose (a_n) is decreasing and null. Then the alternating series $\sum(-1)^{n+1}a_n$ is convergent.

Proof. We first observe that the subsequence (s_{2n}) converges:

$$s_{2n} = \sum_{k=1}^{2n}(-1)^{k+1}a_k = (a_1-a_2)+(a_3-a_4)+...+(a_{2n-1}-a_{2n}) = \sum_{k=1}^{n}(a_{2k-1}-a_{2k}).$$

This is a sum of nonnegative terms. It does not tend to infinity, because it is bounded:

$$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - ... - a_{2n} \leq a_1.$$

Then (s_{2n}) converges to a number ℓ. There remains to show that the whole sequence (s_n) converges to ℓ. For every $\varepsilon > 0$, there exists N such that for all even $m > N$,

$$|s_m - \ell| < \frac{\varepsilon}{2}.$$

If m is odd, we have

$$|s_m - \ell| \leq |s_{m+1} - \ell| + |s_{m+1} - s_m| < \frac{\varepsilon}{2} + |a_{m+1}|.$$

This is less than ε if m is large enough, since the sequence (a_m) tends to 0. ■

Example Since $\left(\frac{1}{n}\right)$ is a decreasing null sequence of this test tells us right away that $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is convergent.

Similarly, $\left(\frac{1}{\sqrt{n}}\right)$ is a decreasing null sequence, therefore $\sum \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \frac{1}{\sqrt{5}} - \ldots$ is convergent.

Pairing terms in a suitable fashion, as in the proof above, one can get the following error bounds.

Exercise 1 Show that, if (a_n) is a decreasing and null sequence, then

$$\left|\sum_{k=N}^{\infty}(-1)^{k+1}a_k\right| \leq a_N.$$

Exercise 2 Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Find a value of N so that $|\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - s| \leq 10^{-6}$.

The Alternating Series test requires that the sequence be decreasing and null, hence it must be non-negative. The next exercise shows that if we relax either the decreasing or null condition then the alternating series may not converge - even if we still insist on the terms being non-negative.

Exercise 3 Find a sequence (a_n) which is non-negative and decreasing but where $\sum(-1)^{n+1}a_n$ is divergent and a sequence (b_n) which is non-negative and null but where $\sum(-1)^{n+1}b_n$ is divergent.

Exercise 4 The series $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \ldots$ converges to $\sin 1$. Explain how to use the series to calculate $\sin 1$ to within an error of 10^{-10}.

Exercise 5 Using the Alternating Series Test where appropriate, show that each of the following series is convergent.

1. $\sum \frac{(-1)^{n+1}n^2}{n^3+1}$
2. $\sum \left(-\frac{1}{2}\right)^n$
3. $\sum \frac{2|\cos\frac{n\pi}{2}|+(-1)^n n}{\sqrt{(n+1)^3}}$
4. $\sum \frac{1}{n}\sin\frac{n\pi}{2}$

9.2 General Series

Series with positive terms are easier because we can attempt to prove that the partial sums (s_n) converge by exploiting the fact that (s_n) is increasing. For a general series $\sum a_n$, we get some information by studying the series of absolute values, $\sum |a_n|$, which involves only positive terms.

Definition
The series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent.

Example The alternating series $\sum \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent because $\sum |\frac{(-1)^{n+1}}{n^2}| = \sum \frac{1}{n^2}$ is convergent.

The series $\sum \frac{(-1)^n}{n}$ is not absolutely convergent because $\sum \frac{1}{n}$ diverges.

The series $\sum \left(-\frac{1}{2}\right)^n$ is absolutely convergent because $\sum \left(\frac{1}{2}\right)^n$ converges.

Exercise 6 Is the series $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$ absolutely convergent?

Exercise 7 For what values of x is the Geometric Series $\sum x^n$ absolutely convergent?

Absolutely convergent series are important for the following reason.

Theorem *Absolute Convergence*
Every absolutely convergent series is convergent.

Proof. Let $s_n = \sum_{i=1}^n a_i$ and $t_n = \sum_{i=1}^n |a_i|$. We know that (t_n) is convergent, hence Cauchy: for every $\varepsilon > 0$, there exists N such that $|t_m - t_n| < \varepsilon$ for all $m, n > N$. We now show that (s_n) is also Cauchy. Let $n > m$.

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| = t_n - t_m < \varepsilon.$$

Then (s_n) is Cauchy, and it converges. ■

Exercise 8 Is the converse of the theorem true: "Every convergent series is absolutely convergent"?

The Absolute Convergence Theorem breathes new life into all the tests we developed for series with non-negative terms: if we can show that $\sum |a_n|$ is convergent, using one of these tests, then we are guarenteed that $\sum a_n$ converges as well.

Exercise 9 Show that the series $\sum \frac{\sin n}{n^2}$ is convergent.

We see that $0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$. Therefore $\sum \frac{|\sin n|}{n^2}$ is convergent by the Comparison Test. It follows that $\sum \frac{\sin n}{n^2}$ is convergent by the Absolute Convergence Theorem.

The Ratio Test can be modified to cope directly with series of mixed terms.

Theorem *Ratio Test*
Suppose $a_n \neq 0$ for all n and $|\frac{a_{n+1}}{a_n}| \to \ell$. Then $\sum a_n$ converges absolutely (and hence converges) if $0 \leq \ell < 1$ and diverges if $\ell > 1$.

Proof. If $0 \le \ell < 1$, then $\sum |a_n|$ converges by the "old" Ratio Test. Therefore $\sum a_n$ converges by the Absolute Convergence Theorem.

If $\ell > 1$, we are guaranteed that $\sum |a_n|$ diverges, but this does not, in itself, prove that $\sum a_n$ diverges (why not?). We have to go back and modify our original proof.

We know that there exists N such that $\frac{|a_{n+1}|}{|a_n|}| \ge 1$ when $n > N$. Then

$$|a_{n+1}| \ge |a_n| \ge |a_{n-1}| \ge ... \ge |a_N| > 0.$$

Therefore the sequence (a_n) does not tend to 0, and $\sum a_n$ diverges by the null sequence test. ■

Example Consider the series $\sum \frac{x^n}{n}$. When $x = 0$ the series is convergent. (Notice that we cannot use the Ratio Test in this case.)

Now let $a_n = \frac{x^n}{n}$. When $x \ne 0$ then $|\frac{a_{n+1}}{a_n}| = |\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}| = \frac{n}{n+1}|x| \to |x|$. Therefore $\sum \frac{x^n}{n}$ is convergent when $|x| < 1$ and divergent when $|x| > 1$, by the Ratio Test.

What if $|x| = 1$? When $x = 1$ then $\sum \frac{x^n}{n} = \sum \frac{1}{n}$ which is divergent. When $x = -1$ then $\sum \frac{x^n}{n} = \sum -\frac{(-1)^{n+1}}{n}$ which is convergent.

Theorem *Ratio Test Variant*
Suppose $a_n \ne 0$ for all n and $|\frac{a_{n+1}}{a_n}| \to \infty$, then $\sum a_n$ diverges.

Exercise 10 Prove this theorem.

Exercise 11 In the next question you will need to use the fact that if a non-negative sequence $(a_n) \to a$ and $a > 0$, then $\left(\sqrt{a_n}\right) \to \sqrt{a}$. Prove this, by first showing that

$$\sqrt{a_n} - \sqrt{a} = \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}.$$

Exercise 12 Determine for which values of x the following series converge and diverge. [Make sure you don't neglect those values for which the Ratio Test doesn't apply.]

1. $\sum \frac{x^n}{n!}$
2. $\sum \frac{n}{x^n}$
3. $\sum n!x^n$
4. $\sum \frac{(2x)^n}{n}$
5. $\sum \frac{(4x)^{3n}}{\sqrt{n+1}}$
6. $\sum(-nx)^n$

9.3 Euler's Constant

Our last aim in this Chapter is to find an explicit formula for the sum of the alternating series:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$$

On the way we shall meet Euler's constant, usually denoted by γ, which occurs in several places in mathematics, especially in number theory.

Euler's Constant

The limit of the sequence

$$D_n=\sum_{i=1}^{n}\frac{1}{i}-\log(n+1)$$

is called Euler's Constant and is usually denoted by γ (gamma). Its value has been computed to over 200 decimal places. To 14 decimal places, it is 0.57721566490153. No-one knows whether γ is rational or irrational.

Exercise 13 Let $D_n=\sum_{i=1}^{n}\frac{1}{i}-\int_1^{n+1}\frac{1}{x}dx=\sum_{i=1}^{n}\frac{1}{i}-\log(n+1)$.

1. Using Figure 9.1, draw the areas represented by D_n.
2. Show that (D_n) is increasing.
3. Show that (D_n) is bounded - and hence convergent.

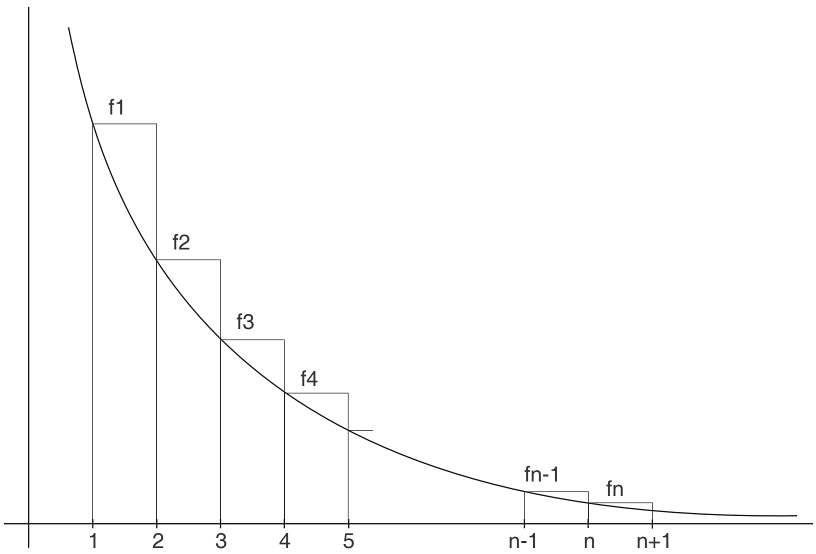

Figure 9.1: Calculating an upper bound of an integral.

The limit of the sequence $D_n=\sum_{i=1}^{n}\frac{1}{i}-\log(n+1)$ is called Euler's Constant and is usually denoted by γ.

Exercise 14 Show that $\sum_{i=1}^{2n-1}\frac{(-1)^{i+1}}{i}=\log 2+D_{2n-1}-D_{n-1}$. Hence evaluate $\sum\frac{(-1)^{n+1}}{n}$.

Hint: First, derive the following identity:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2n-1}$$
$$=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n-1}-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\frac{1}{n-1}\right)$$

9.4 * Application - Stirling's Formula *

Using the alternating series test we can improve the approximations to $n!$ that we stated in workbook 4. Take a look at what we did there: we obtained upper and lower bounds to $\log(n!)$ by using block approximations to the integral of $\int_1^n \log x dx$. To get a better approximation we use the approximation in Figure 9.2.

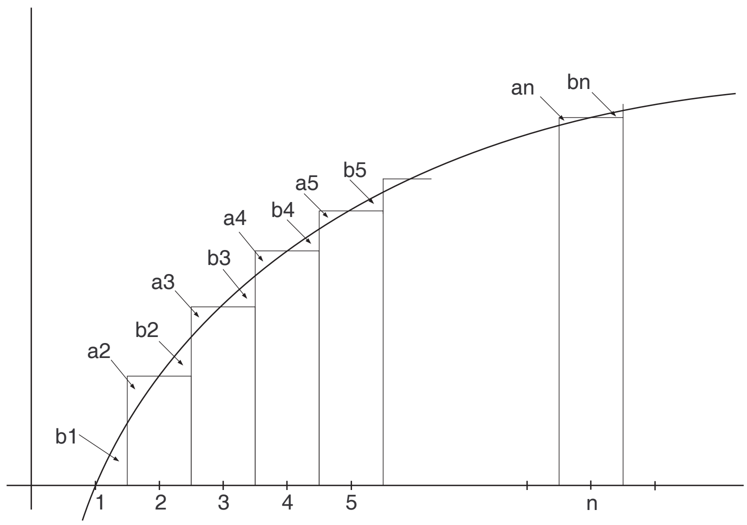

Figure 9.2: Approximating the integral by the mid point.

Now the area of the blocks approximates $\int_1^n \log x dx$ except that there are small triangular errors below the graph (marked as $b_1, b_2, b_3, \dots$) and small triangular errors above the graph (marked as $a_2, a_3, a_4, \dots$).

Note that $\log n! = \log 2 + \log 3 + \cdots + \log n =$ area of the blocks.

Exercise 15 Use the above diagram to show:

$$\log n! - \left(n + \tfrac{1}{2}\right) \log n + n = 1 - b_1 + a_2 - b_2 + a_3 - b_3 + \cdots - b_{n-1} + a_n$$

[Hint: $\int_1^n \log x dx = n \log n - n + 1$]

The curve $\log x$ is concave and it seems reasonable (and can be easily proved - try for yourselves), that $a_n \geq b_n \geq a_{n+1}$ and $(a_n) \to 0$.

Exercise 16 Assuming that these claims are true, explain why $(s_n) = (1 - b_1 + a_2 - b_2 + \cdots + a_n - b_n)$ converges.

This proves that $\log n! = (n + \frac{1}{2}) \log n - n + \Sigma_n$ where Σ_n tends to a constant as $n \to \infty$. Taking exponentials we obtain:

$$n! \simeq \text{constant} \cdot n^n e^{-n} \sqrt{n}$$

What is the constant? This was identified with only a little more work by the mathematician James Stirling. Indeed, he proved that:

$$\frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \to 1 \quad \text{as } n \to \infty$$

a result known as Stirling's formula.

Check Your Progress

By the end of this chapter you should be able to:

- Use and justify the Alternating Series Test: If (a_n) is a decreasing null sequence then $\sum(-1)^{n+1}a_n$ is convergent.
- Use the proof of Alternating Series Test to establish error bounds.
- Say what it means for a series to be *absolutely convergent* and give examples of such series.
- Prove that an absolutely convergent series is convergent, but that the converse is not true.
- Use the modified Ratio Test to determine the convergence or divergence of series with positive and negative terms.
- Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$.