Power Rule

Let $x > 0$, $y > 0$

Claim $x < y \iff x^n < y^n$

Proof Let $S_n = \sum_{k=0}^{n-1} x^k y^{n-1-k} > 0$

$$(x-y) S_n = \sum_{k=0}^{n-1} x^{k+1} y^{n-1-k} - \sum_{k=0}^{n-1} x^k y^{n-k}$$

$$= \sum_{k=1}^{n} x^k y^{n-k} - \sum_{k=0}^{n-1} x^k y^{n-k}$$

$$= x^n - y^n + \sum_{k=1}^{n-1} x^k y^{n-k} - \sum_{k=1}^{n-1} x^k y^{n-k} = x^n - y^n$$

Proved that $x^n - y^n = (x-y) S_n$, where $S_n > 0$

Therefore, $x^n - y^n > 0 \iff x - y > 0$

Done.

Let $a_1, a_2, \dots, a_n \geq 0$, $n \in \mathbb{N}$

Let $A_n = \frac{1}{n} \sum_{k=1}^{n} a_k$ (Arithm. mean)

Let $G_n = \left(\prod_{k=1}^{n} a_k \right)^{\frac{1}{n}}$ (geom. mean)

Claim: $A_n \geq G_n$

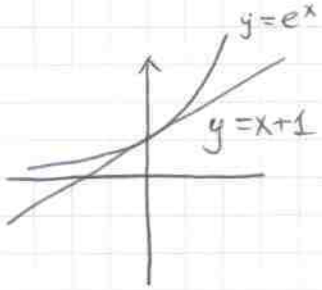
Proof (Pólya):

If $a_1 = a_2 = \dots = a_n$ we are done.

Assume that not all a's are zero $\Rightarrow$

$A_n > 0$

Note that $\forall \; x \in \mathbb{R}$



$$e^x \geq x + 1 \iff e^{x-1} \geq x \Rightarrow$$

$$e^{\frac{a_k}{A_n} - 1} \geq a_k \qquad k = 1, \dots, n$$

$$\Downarrow$$

$$\prod_{k=1}^{n} e^{\frac{a_k}{A_n} - 1} \geq \prod_{k=1}^{n} (a_k / A_n)$$

$$e^{\sum_{k=1}^{n} \left(\frac{a_k}{A_n} - 1 \right)} \geq G_n^n \frac{1}{A_n^n}$$

$$e^{\frac{n A_n}{A_n} - n} \geq \frac{G_n^n}{A_n^n}$$

$1 \geq \frac{G_n}{}$ Done