Riemann series theorem

In mathematics, the **Riemann series theorem** (also called the **Riemann rearrangement theorem**), named after 19th-century German mathematician Bernhard Riemann, says that if an infinite series is conditionally convergent, then its terms can be arranged in a permutation so that the series converges to any given value, or even diverges.

Definitions

A series $\sum_{n=1}^{\infty} a_n$ converges if there exists a value ℓ such that the sequence of the partial sums

$$\{S_1,\ S_2,\ S_3,\dots\}, \quad S_n = \sum_{k=1}^{n} a_k,$$

converges to ℓ . That is, for any $\varepsilon > 0$, there exists an integer N such that if $n \geq N$, then

$$|S_n - \ell| \leq \ \varepsilon.$$

A series converges conditionally if the series $\sum_{n=1}^{\infty} a_n$ converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

A permutation is simply a bijection from the set of positive integers to itself. This means that if σ is a permutation, then for any positive integer b , there exists exactly one positive integer a such that $\sigma(a) = b$. In particular, if $x \neq y$, then $\sigma(x) \neq \sigma(y)$.

Statement of the theorem

Suppose that

$$\{a_1,\ a_2,\ a_3,\dots\}$$

is a sequence of real numbers, and that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Let M be a real number. Then there exists a permutation $\sigma(n)$ of the sequence such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M.$$

There also exists a permutation $\sigma(n)$ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \infty.$$

The sum can also be rearranged to diverge to $-\infty$ or to fail to approach any limit, finite or infinite.

Examples

Changing the sum

The alternating harmonic series is a classic example of a conditionally convergent series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is convergent, while

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right|$$

is the ordinary harmonic series, which diverges. Although in standard presentation the alternating harmonic series converges to ln(2), its terms can be arranged to converge to any number, or even to diverge. One instance of this is as follows. Begin with the series written in the usual order,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

and rearrange the terms:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}+\cdots$$

where the pattern is: the first two terms are 1 and −1/2, whose sum is 1/2. The next term is −1/4. The next two terms are 1/3 and −1/6, whose sum is 1/6. The next term is −1/8. The next two terms are 1/5 and −1/10, whose sum is 1/10. In general, the sum is composed of blocks of three:

$$\frac{1}{2k-1}-\frac{1}{2(2k-1)}-\frac{1}{4k},\quad k=1,2,\dots.$$

This is indeed a rearrangement of the alternating harmonic series: every odd integer occurs once positively, and the even integers occur once each, negatively (half of them as multiples of 4, the other half as twice odd integers). Since

$$\frac{1}{2k-1}-\frac{1}{2(2k-1)}=\frac{1}{2(2k-1)},$$

this series can in fact be written:

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}+\cdots+\frac{1}{2(2k-1)}-\frac{1}{2(2k)}+\cdots$$
$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}+\cdots\right)=\frac{1}{2}\ln(2)$$

which is half the usual sum.

Getting an arbitrary sum

An efficient way to recover and generalize the result of the previous section is to use the fact that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\gamma+\ln n+o(1),$$

where γ is the Euler–Mascheroni constant, and where the notation o(1) denotes a quantity that depends upon the current variable (here, the variable is n) in such a way that this quantity goes to 0 when the variable tends to infinity.

It follows that the sum of q even terms satisfies

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2q}=\frac{1}{2}\gamma+\frac{1}{2}\ln q+o(1),$$

and by taking the difference, one sees that the sum of p odd terms satisfies

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2p-1}=\frac{1}{2}\gamma+\frac{1}{2}\ln p+\ln 2+o(1).$$

Suppose that two positive integers a and b are given, and that a rearrangement of the alternating harmonic series is formed by taking, in order, a positive terms from the alternating harmonic series, followed by b negative terms, and repeating this pattern at infinity (the alternating series itself corresponds to $a = b = 1$, the example in the preceding section corresponds to $a = 1$, $b = 2$):

$$1+\frac{1}{3}+\cdots+\frac{1}{2a-1}-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2b}+\frac{1}{2a+1}+\cdots+\frac{1}{4a-1}-\frac{1}{2b+2}-\cdots$$

Then the partial sum of order $(a+b)n$ of this rearranged series contains $p = a\,n$ positive odd terms and $q = b\,n$ negative even terms, hence

$$S_{(a+b)n} = \frac{1}{2}\ln p + \ln 2 - \frac{1}{2}\ln q + o(1) = \frac{1}{2}\ln(a/b) + \ln 2 + o(1).$$

It follows that the sum of this rearranged series is

$$\frac{1}{2}\ln(a/b) + \ln 2 = \ln\left(2\sqrt{a/b}\right).$$

Suppose now that, more generally, a rearranged series of the alternating harmonic series is organized in such a way that the ratio p_n / q_n between the number of positive and negative terms in the partial sum of order n tends to a positive limit r. Then, the sum of such a rearrangement will be

$$\ln\left(2\sqrt{r}\right),$$

and this explains that any real number x can be obtained as sum of a rearranged series of the alternating harmonic series: it suffices to form a rearrangement for which the limit r is equal to $e^{2x} / 4$.

Proof

For simplicity, this proof assumes first that $a_n \neq 0$ for every n. The general case requires a simple modification, given below. Recall that a conditionally convergent series of real terms has both infinitely many negative terms and infinitely many positive terms. First, define two quantities, a_n^+ and a_n^- by:

$$a_n^+ = \frac{a_n + |a_n|}{2}, \quad a_n^- = \frac{a_n - |a_n|}{2}.$$

That is, the series $\sum_{n=1}^{\infty} a_n^+$ includes all a_n positive, with all negative terms replaced by zeroes, and the series $\sum_{n=1}^{\infty} a_n^-$ includes all a_n negative, with all positive terms replaced by zeroes. Since $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, both the positive and the negative series diverge. Let M be a positive real number. Take, in order, just enough positive terms a_n^+ so that their sum exceeds M. Suppose we require p terms – then the following statement is true:

$$\sum_{n=1}^{p-1} a_n^+ \leq M < \sum_{n=1}^{p} a_n^+.$$

This is possible for any $M > 0$ because the partial sums of a_n^+ tend to $+\infty$. Discarding the zero terms one may write

$$\sum_{n=1}^{p} a_n^+ = a_{\sigma(1)} + \cdots + a_{\sigma(m_1)}, \quad a_{\sigma(j)} > 0, \quad \sigma(1) < \ldots < \sigma(m_1) = p.$$

Now we add just enough negative terms a_n^-, say q of them, so that the resulting sum is less than M. This is always possible because the partial sums of a_n^- tend to $-\infty$. Now we have:

$$\sum_{n=1}^{p} a_n^+ + \sum_{n=1}^{q} a_n^- < M \leq \sum_{n=1}^{p} a_n^+ + \sum_{n=1}^{q-1} a_n^-.$$

Again, one may write

$$\sum_{n=1}^{p} a_n^+ + \sum_{n=1}^{q} a_n^- = a_{\sigma(1)} + \cdots + a_{\sigma(m_1)} + a_{\sigma(m_1+1)} + \cdots + a_{\sigma(n_1)},$$

with

$$\sigma(m_1 + 1) < \ldots < \sigma(n_1) = q.$$

Note that σ is injective, and that 1 belongs to the range of σ, either as image of 1 (if $a_1 > 0$), or as image of $m_1 + 1$ (if $a_1 < 0$). Now repeat the process of adding just enough positive terms to exceed M, starting with $n = p + 1$, and then adding just enough negative terms to be less than M, starting with $n = q + 1$. Extend σ in an injective manner, in order to cover all terms selected so far, and observe that a_2 must have been selected now or before, thus 2 belongs to the range of this extension. The process will have infinitely many such "*changes of direction*". One eventually obtains a rearrangement $\sum a_{\sigma(n)}$. After the first change of direction, each partial sum of $\sum a_{\sigma(n)}$ differs from M by at most the absolute value $a^+_{p_j}$ or $|a^-_{q_j}|$ of the term that appeared at the latest change of direction. But $\sum a_n$ converges, so as n tends to infinity, each of a_n, $a^+_{p_j}$ and $a^-_{q_j}$ go to 0. Thus, the partial sums of $\sum a_{\sigma(n)}$ tend to M, so the following is true:

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M.$$

The same method can be used to show convergence to M negative or zero.

One can now give a formal inductive definition of the rearrangement σ, that works in general. For every integer $k \geq 0$, a finite set A_k of integers and a real number S_k are defined. For every $k > 0$, the induction defines the value $\sigma(k)$, the set A_k consists of the values $\sigma(j)$ for $j \leq k$ and S_k is the partial sum of the rearranged series. The definition is as follows:

- For $k = 0$, the induction starts with A_0 empty and $S_0 = 0$.
- For every $k \geq 0$, there are two cases: if $S_k \leq M$, then $\sigma(k+1)$ is the smallest integer $n \geq 1$ such that n is not in A_k and $a_n \geq 0$; if $S_k > M$, then $\sigma(k+1)$ is the smallest integer $n \geq 1$ such that n is not in A_k and $a_n < 0$. In both cases one sets

$$A_{k+1} = A_k \cup \{\sigma(k+1)\}; \quad S_{k+1} = S_k + a_{\sigma(k+1)}.$$

It can be proved, using the reasonings above, that σ is a permutation of the integers and that the permuted series converges to the given real number M.

Generalization

Given a converging series $\sum a_n$ of complex numbers, several cases can occur when considering the set of possible sums for all series $\sum a_{\sigma(n)}$ obtained by rearranging (permuting) the terms of that series:

- the series $\sum a_n$ may converge unconditionally; then, all rearranged series converge, and have the same sum: the set of sums of the rearranged series reduces to one point;
- the series $\sum a_n$ may fail to converge unconditionally; if S denotes the set of sums of those rearranged series that converge, then, either the set S is a line L in the complex plane **C**, of the form

$$L = \{a + tb : t \in \mathbf{R}\}, \quad a, b \in \mathbf{C},\ b \neq 0,$$

  or the set S is the whole complex plane **C**.

More generally, given a converging series of vectors in a finite dimensional real vector space E, the set of sums of converging rearranged series is an affine subspace of E.

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[1] http://mathworld.wolfram.com/RiemannSeriesTheorem.html

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