Analysis III

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Part I

Integration

1 The Integral for Step Functions

1.1 Definition of the Integral for Step Functions

Motivation: "Integrals = Areas Under Curves"

Consider constant functions of the form $f_c : x \mapsto f_0 \in \mathbb{R}$ for $x \in [a, b]$. Let D be the area under the line at f_0 from a to b. Then it makes sense to define

$$\begin{aligned}\int_a^b f_c(x)\ dx &:= \text{sign}(f_0) \cdot \text{Area}(D) \\ &= \text{sign}(f_0) \cdot (|f_0| \cdot (b - a)) \\ &= f_0 \cdot (b - a)\end{aligned}$$

Now this definition will be extended to piecewise constant or step functions.

Definition 1: $\varphi : [a, b] \to \mathbb{R}$ is called a *step function* if there exists a finite set $P \subset [a, b]$ with $a, b \in P$ such that φ is constant on the open sub-intervals of $[a, b] \setminus P$. P is called a *partition* of $[a, b]$. More concretely, $P = \{a = p_0, p_1, p_2, \ldots, p_{k-1}, p_k = b\}$, with $p_0 < p_1 < p_2 < \ldots < p_{k-1} < p_k$, and $\varphi\Big|_{(p_{i-1}, p_i)} = c_i$ (an arbitrary constant), for $i = 1, 2, \ldots, k-1, k$.

Notes: If ϕ is a step function then there are many partitions of $[a, b]$ such that ϕ is constant on the open sub-intervals defined by the partition, call one P. There exists at least one other point greater than a in P (since both a and b belong to P by definition), call the first p. Then ϕ is also constant on the open intervals of $[a, b] \setminus (P \cup \{n\})$, where $a < n < p$. Let Q be a partition of $[a, b]$, with $Q \supset P$. Then Q is called a *refinement* of P (so $(P \cup \{n\})$ is a refinement of P). The minimal common refinement of P and Q is denoted by $P \cup Q$.

If ϕ is a step function on $[a, b]$, then any partition $P = \{a = p_0, p_1, p_2, \ldots, p_{k-1}, p_k = b\}$, with $p_0 < p_1 < p_2 < \ldots < p_{k-1} < p_k$, such that $\phi\Big|_{(p_{i-1}, p_i)} = c_i$ (an arbitrary constant), for $i = 1, 2, \ldots, k-1, k$, is called *compatible with*, or *adapted to*, ϕ.

Proposition 1: Let $S[a, b]$ be the space of step functions on $[a, b]$. If $f, g \in S[a, b]$, then $\forall \mu, \lambda \in \mathbb{R}$, $(\lambda f + \mu g) \in S[a, b]$ (so $S[a, b]$ is closed under taking finite linear combinations of its elements).

Proof: $f \in S[a, b]$, so $\exists P = \{a = p_0, p_1, p_2, \ldots, p_{k-1}, p_k = b\}$, with $p_0 < p_1 < p_2 < \ldots < p_{k-1} < p_k$, which is compatible with f, and $g \in S[a, b]$ so $\exists Q = \{a = q_0, q_1, q_2, \ldots, q_{l-1}, q_l = b\}$, with $q_0 < q_1 < q_2 < \ldots < q_{l-1} < q_l$, which is compatible with g. Consider $P \cup Q = \{a = r_0, r_1, r_2, \ldots, r_{m-1}, r_m = b\}$, with $r_0 < r_1 < r_2 < \ldots < r_{m-1} < r_m$. Then this partition is compatible with $(\lambda f + \mu g)$. It remains to show that $(\lambda f + \mu g)\Big|_{(r_{i-1}, r_i)} = c_i$ (an arbitrary constant), for $i = 1, 2, \ldots, m-1, m$.

$P \subset (P \cup Q)$, so $(r_{i-1}, r_i) \subset (p_{a-1}, p_a)$ for some a, and $Q \subset (P \cup Q)$, so $(r_{i-1}, r_i) \subset (q_{b-1}, q_b)$ for some b. Then, since $f\Big|_{(p_{a-1}, p_a)}$ is constant, $f\Big|_{(r_{i-1}, r_i)}$ is constant, and since $g\Big|_{(q_{b-1}, q_b)}$ is constant, $g\Big|_{(r_{i-1}, r_i)}$ is constant, so $(\lambda f + \mu g)\Big|_{(r_{i-1}, r_i)}$ is constant. Since the choice of i was arbitrary, this proves the proposition. ■

Notes: This demonstrates that $S[a,b]$ is a vector space over $\mathbb{R}$. Define $C[a,b]$ as the space of continuous functions on $[a,b]$, and define $B[a,b]$ as the space of bounded functions on $[a,b]$. These are also vector spaces. It was demonstrated in Analysis II that $C[a,b] \subset B[a,b]$, and by definition $S[a,b] \subset B[a,b]$.

Definition 2: For $\phi \in S[a,b]$, let $P = \{p_i\}_{i=0}^k$ be a partition compatible with ϕ. Define (temporarily) $\phi_i := \phi\Big|_{(p_{i-1},p_i)}$. Then $\int_a^b : S[a,b] \to \mathbb{R}$ is defined as

$$\int_a^b \phi := \sum_{i=1}^k \phi_i(p_i - p_{i-1})$$

Remark: If $\phi \equiv \phi_0 \in \mathbb{R}$, then $P = \{a,b\}$, so $\int_a^b \phi = \phi_0(b-a)$, which is precisely the area described in the motivation.

Lemma 2: The value of $\int_a^b \phi$ does not depend on the choice of partition compatible with $[a,b]$.

Proof: For any $\phi \in S[a,b]$, let P and Q be partitions compatible with $[a,b]$. Suppose there are k points in Q that are not in P. Denote them $n_1, n_2, \ldots, n_k$. Temporarily let $\int_a^b \phi\Big|_P$ be the integral of ϕ from a to b defined by the partition P which is compatible with $[a,b]$.

Now, consider $P \cup \{n_1\}$. Since Q is compatible with $[a,b]$, we know that $n_1 \in [a,b]$, so $\exists i \in \mathbb{N}$ such that $n_1 \in (p_{i-1}, p_i)$ for some $p_{i-1}, p_i \in P$. ϕ is constant on (p_{i-1}, p_i), and so is also constant on (p_{i-1}, n_1) and (n_1, p_i), so $P \cup \{n_1\}$ is compatible with ϕ. By definition

$$\int_a^b \phi\Big|_{P\cup\{n_1\}} - \int_a^b \phi\Big|_P = \phi_i(n_1 - p_{i-1}) + \phi_i(p_i - n_1) - \phi_i(p_i - p_{i-1}) = 0, \text{ so } \int_a^b \phi\Big|_{P\cup\{n_1\}} = \int_a^b \phi\Big|_P$$

The proof is completed by induction, since there are a finite number of points. This is left as an exercise. ■

1.2 Properties of the Integral for Step Functions

1. **Proposition 3 - Additivity:** For any $\phi \in S[a,b]$, $\forall v \in (a,b)$,

$$\int_a^b \phi = \int_a^v \phi + \int_v^b \phi$$

2. **Proposition 4 - Linearity:** For any $\phi, \psi \in S[a,b]$, $\forall \lambda, \mu \in \mathbb{R}$,

$$\int_a^b \lambda\phi + \mu\psi = \lambda \int_a^b \phi + \mu \int_a^b \psi$$

3. **Proposition 5 - The Fundamental Theorem of Calculus:** Let $\phi \in S[a,b]$, and let P be a partition of $[a,b]$ compatible with ϕ. Consider $I : [a,b] \to \mathbb{R}$, $I : x \mapsto \int_a^x \phi$. Then
   (I) I is continuous on $[a,b]$.
   (II) I is differentiable on $\bigcup_{i=1}^k (p_{i-1}, p_i)$, and, $\forall x \in \bigcup_{i=1}^k (p_{i-1}, p_i)$, $I'(x) = \phi(x)$.

Proof:

1. Let $P = \{p_i\}_{i=0}^k$ be a partition compatible with $\phi \in S[a,b]$. Then there are two cases:
   (i) If $v \in P$, then by definition

$$\int_a^b \phi = \sum_{i=1}^k \phi_i(p_i - p_{i-1}) = \sum_{i=1}^v \phi_i(p_i - p_{i-1}) + \sum_{i=v+1}^k \phi_i(p_i - p_{i-1}) = \int_a^v \phi + \int_v^b \phi$$

   (ii) If $v \notin P$, then $\exists i \in \mathbb{N}$, $1 \le i \le k$, such that $v \in (p_{i-1}, p_i)$. Then $\{a = p_0, \dots, p_{i-1}, v\}$, $\{v, p_i, \dots, p_k = b\}$ are partitions compatible with ϕ, so $\phi\Big|_{(a,v)} \in S[a,v]$, $\phi\Big|_{(v,b)} \in S[v,b]$, so their integrals are well-defined on those intervals. Then

$$\begin{aligned}\int_a^b \phi &= \sum_{j=1}^k \phi_j(p_j - p_{j-1}) \\ &= \phi_1(p_1 - p_0) + \phi_2(p_2 - p_1) + \ldots + \phi_i(p_i - p_{i-1}) + \ldots + \phi_k(p_k - p_{k-1}) \\ &= \phi_1(p_1 - p_0) + \ldots + \phi_i(p_i + v - v - p_{i-1}) + \ldots + \phi_k(p_k - p_{k-1}) \\ &= \phi_1(p_1 - p_0) + \ldots + \phi_i(v - p_{i-1}) + \phi_i(p_i - v) + \ldots + \phi_k(p_k - p_{k-1}) \\ &= \left[\sum_{j=1}^{i-1} \phi_j(p_j - p_{j-1}) + \phi_i(v - p_{i-1})\right] + \left[\phi_i(p_i - v) + \sum_{j=i+1}^k \phi_j(p_j - p_{j-1})\right] \\ &= \int_a^v \phi + \int_v^b \phi \quad \blacksquare\end{aligned}$$

2. Let P be a partition compatible with $\phi \in S[a,b]$, and let Q be a partition compatible with $\psi \in S[a,b]$. Then, as was proved in the previous section, $P \cup Q$ is compatible with both ϕ and ψ. Let $P \cup Q = \{a = x_0, x_1, \dots, x_{k-1}, x_k = b\}$, with $a = x_0 < x_1 < \ldots < x_{k-1} < x_k = b$. Then $\phi\Big|_{(x_{i-1}, x_i)} = \phi_i$ and $\psi\Big|_{(x_{i-1}, x_i)} = \psi_i$, so $(\lambda\phi + \mu\psi)\Big|_{(x_{i-1}, x_i)} = (\lambda\phi + \mu\psi)_i$, so $P \cup Q$ is compatible with $\lambda\phi + \mu\psi$. Then

$$\begin{aligned}\int_a^b \lambda\phi + \mu\psi &= \sum_{j=1}^k (\lambda\phi + \mu\psi)_j (x_j - x_{j-1}) = \sum_{j=1}^k (\lambda\phi_j + \mu\psi_j)(x_j - x_{j-1}) \\ &= \lambda \sum_{j=1}^k \phi_j(x_j - x_{j-1}) + \mu \sum_{j=1}^k \psi_j(x_j - x_{j-1}) = \lambda \int_a^b \phi + \mu \int_a^b \psi \quad \blacksquare\end{aligned}$$

3. Let $P = \{p_i\}_{i=0}^k$ be a partition compatible with $\phi \in S[a,b]$. Choose $x \in \bigcup_{j=1}^k (p_{j-1}, p_i)$, then $\exists i \in \mathbb{N}$, $1 \leq i \leq k$ such that $x \in (p_{i-1}, p_i)$. Then

$$I(x) = \int_a^x \phi = \int_a^{p_{i-1}} \phi + \int_{p_{i-1}}^x \phi = I(p_{i-1}) + \phi_i(x - p_{i-1}) = \phi_i x + \text{constant}$$

so I is continuous on $\bigcup_{j=1}^k (p_{j-1}, p_i)$ and $I'(x) = \phi_i = \phi(x)$. Furthermore, $\lim_{x \downarrow p_{i-1}} I(x) = I(p_{i-1})$, so I is right continuous on $[a,b)$. Similarly, $\lim_{x \uparrow p_i} I(x) = I(p_{i-1}) + \phi_i(p_i - p_{i-1}) = \int_a^{p_{i-1}} \phi + \phi_i(p_i - p_{i-1}) = \int_a^{p_{i-1}} \phi + \int_{p_{i-1}}^{p_i} \phi = \int_a^{p_i} \phi = I(p_i)$, so I is left continuous on $(a,b]$, so I is continuous on $[a,b]$. ■

Definition 3: Let $f \in B[a,b]$. Then the *supremum norm* (or *sup norm*) of f, denoted by $\|f\|_\infty$, with $\|\cdot\|_\infty : B[a,b] \to \mathbb{R}$ is defined as

$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|$$

Proposition 6: Let $\phi \in S[a,b]$ be bounded between $m, M \in \mathbb{R}$, so $m \leq \phi(x) \leq M$ $\forall x \in [a,b]$. Then

$$m(b-a) \leq \int_a^b \phi \leq M(b-a), \text{ in particular } \left| \int_a^b \phi \right| \leq \|\phi\|_\infty (b-a).$$

Proof: For $\phi \in S[a,b]$, choose a compatible partition P. Now

$$\int_a^b \phi = \sum_{i=1}^k \phi_i(p_i - p_{i-1})$$

then

$$\sum_{i=1}^k m(p_i - p_{i-1}) \leq \sum_{i=1}^k \phi_i(p_i - p_{i-1}) \leq \sum_{i=1}^k M(p_i - p_{i-1})$$

so, due to telescoping sums on the upper and lower bounds,

$$m(b-a) \leq \sum_{i=1}^k \phi_i(p_i - p_{i-1}) \leq M(b-a), \text{ so } m(b-a) \leq \int_a^b \phi \leq M(b-a).$$

If $\|\phi\|_\infty$ is given, then, by definition, $-\|\phi\|_\infty \leq \phi(x) \leq \|\phi\|_\infty$, so using the same argument as above, $-\|\phi\|_\infty (b-a) \leq \int_a^b \phi \leq \|\phi\|_\infty (b-a)$, so $\left|\int_a^b \phi\right| \leq \|\phi\|_\infty (b-a)$. ■

Note: So far $\int_a^b \phi$ has been defined for $b > a$. Define, for $a > b$, $\int_a^b \phi := -\int_a^b \phi$. Then the bound proved in the previous proposition becomes $\left|\int_a^b \phi\right| \leq \|\phi\|_\infty |b-a|$. It is a simple exercise to demonstrate that such an extension preserves the three properties of the integral for step functions proved previously.

2 The Integral for Regulated Functions

2.1 Definition of the Integral for Regulated Functions

Definition 4: A function f on $[a,b]$ is called a *regulated function* if, $\forall\varepsilon > 0$, $\exists\phi \in S[a,b]$ such that $\|\phi - f\|_\infty < \varepsilon$, or, alternatively, if, $\forall\varepsilon > 0$, $\left|f(x) - \phi(x)\right| < \varepsilon\ \forall x \in [a,b]$. We denote the set of regulated functions on $[a,b]$ by $R[a,b]$.

Definition 4.1: A sequence $(\phi_n)_{n\geq 1} \subset S[a,b]$ is said to *converge uniformly* to a function $f : [a,b] \to \mathbb{R}$ if $\lim_{n\to\infty}\|f - \phi_n\|_\infty = 0$.

Lemma : $f : [a,b] \to \mathbb{R}$ is regulated iff $\exists(\phi_n)_{n\geq 1} \subset S[a,b]$ such that, as $n \to \infty$, (ϕ_n) converges uniformly to f.

Proof: The proof is routine and is left as an exercise. ■

Definition 4.2: A function $f : A \subset \mathbb{R} \to \mathbb{R}$ is called *uniformly continuous* on A if, $\forall\varepsilon > 0$, $\exists\delta$ such that $\forall x, y \in A, |x - y| < \delta$ and $\left|f(x) - f(y)\right| < \varepsilon$.

Note: In general, uniform continuity implies continuity, but not the other way around. For example, $f : (0,1] \to \mathbb{R}$, $f : x \mapsto \frac{1}{x}$, is continuous, but not uniformly continuous. However, on closed intervals, the two are equivalent.

Lemma: For a function $f : A \to \mathbb{R}$, if $A = [a,b]$, continuity implies uniform continuity.

Proof: Suppose f is continuous on $[a,b]$ but not uniformly continuous. Then $\exists\varepsilon > 0$ such that $\forall\delta > 0$, $\exists x, y \in [a,b]$ such that $|x - y| < \delta$ but $\left|f(x) - f(y)\right| \geq \varepsilon$. Taking $\delta_n = \frac{1}{n} \to 0$ as $n \to \infty$ yields two sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset [a,b]$ with $|x_n - y_n| < \delta_n$. Since $(x_n)_{n\geq 1}$ is bounded, it converges by the Bolzano-Weierstrass Theorem, so $\exists(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ which converges in $[a,b]$, since $[a,b]$ is closed. So $\lim_{k\to\infty} x_{n_k} = u \in [a,b]$. Choose $(y_{n_k})_{k\geq 1}$, which converges by a similar argument, and choosing $\delta = \frac{1}{n_k}$ demonstrates that $\lim_{k\to\infty}\left|x_{n_k} - y_{n_k}\right| = 0$, so $\lim_{k\to\infty} y_{n_k} = u$. But f is continuous, so $\lim_{k\to\infty} f(x_{n_k}) = f(u) = \lim_{k\to\infty} f(y_{n_k})$, which contradicts the condition that $|x - y| < \delta$ but $\left|f(x) - f(y)\right| \geq \varepsilon$, so f is uniformly continuous on $[a,b]$. ■

Proposition 7: If a function $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then it is also regulated on $[a,b]$.

Proof: For any $\varepsilon > 0$, let $P = \{p_i\}_{i=0}^k$ be a partition of $[a,b]$ such that $|p_i - p_{i-1}| < \delta$, where δ is such that $|x - y| < \delta \implies \left|f(x) - f(y)\right| < \varepsilon$. Such a δ exists because f is continuous and so uniformly continuous on $[a,b]$. Let $\phi \in S[a,b]$ be such that $\begin{cases} \phi\Big|_{[p_{i-1},p_i)} = f(p_{i-1}) & \text{for } i = 1,\ldots,k \\ \phi(b) = f(b) \end{cases}$. Now, $\forall x \in [a,b)$, $\exists j$ such that $x \in [p_{j-1}, p_j)$, then $\left|f(x) - \phi(x)\right| = \left|f(x) - f(p_{j-1})\right| < \varepsilon$ since $\left|x - p_{j-1}\right| < \delta$. Also, $\left|f(b) - \phi(b)\right| = 0$, so, $\forall x \in [a,b]$, $\left|f(x) - \phi(x)\right| < \varepsilon$, so f is regulated by definition. ■

Exercise: Prove that any monotone function is regulated.

Proposition 8: For any $f, g \in R[a,b]$ and $\lambda, \mu \in \mathbb{R}$, $\lambda f + \mu g \in R[a,b]$.

Proof: First prove that $\forall f, g \in B[a,b]$, $\forall \lambda \in \mathbb{R}$, $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ and $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$. The remainder of the proof is left as an exercise. ■

Definition 5: Let $f \in R[a,b]$. Then $\int_a^b : R[a,b] \to \mathbb{R}$ is defined as

$$\int_a^b f := \lim_{n\to\infty} \int_a^b \phi_n$$

where $(\phi_n)_{n\geq 1} \subset S[a,b]$ converges to f uniformly as $n \to \infty$.

Proposition 9: $\lim_{n\to\infty} \int_a^b \phi_n$ exists and is independent of the choice of $(\phi_n)_{n\geq 1} \subset S[a,b]$, which converges to $f \in R[a,b]$ uniformly as $n \to \infty$.

Proof: Let $f \in R[a,b]$ and $(\phi_n)_{n\geq 1}, \ (\psi_n)_{n\geq 1} \subset S[a,b]$ converge to f uniformly as $n \to \infty$. (ϕ_n) converges to f uniformly, so for any $\varepsilon > 0$, eventually there will be an n such that $\|\phi_n - f\|_\infty < \frac{\varepsilon}{2(b-a)}$, so eventually there will be n and m, when sufficiently large, such that $\|\phi_n - \phi_m\|_\infty = \big\|(\phi_n - f) + (f - \phi_m)\big\|_\infty$ $\leq \|\phi_n - f\|_\infty + \|f - \phi_m\|_\infty \leq \frac{\varepsilon}{b-a}$. Then $\left|\int_a^b \phi_n - \int_a^b \phi_m\right| = \left|\int_a^b (\phi_n - \phi_m)\right| \leq \|\phi_n - \phi_m\|_\infty (b-a) < \varepsilon$, so $(\int_a^b \phi_n)_{n\geq 1}$ is Cauchy, so the limit exists.

Let $\omega_n = (\phi_n, \ \psi_n)_{n-1} = (\phi_1, \ \psi_1, \ \phi_2, \ \psi_2, \ \ldots) \subset S[a,b]$. Then ω_n converges to f uniformly as $n \to \infty$, the proof of this fact is routine and is left as an exercise. $\lim_{n\to\infty} \int_a^b \omega_n$ exists, so the subsequences $\int_a^b \phi_n$ and $\int_a^b \psi_n$ must converge to the same limit $\int_a^b f$, so $\int_a^b \phi_n = \int_a^b \psi_n$. ■

2.2 Properties of the Integral for Regulated Functions

Proposition 10: For any $f \in R[a,b], u \in [a,b]$,

$$\int_a^b f = \int_a^u f + \int_u^b f$$

Proof: f is regulated, so $\exists (\phi_n)_{n\geq 1} \subset S[a,b]$ which converges uniformly to f. Then $(\phi_n)\Big|_{[a,u]}$ converges uniformly to $f\Big|_{[a,u]}$, since restrictions are regulated. The same holds for $(\phi_n)\Big|_{[u,b]}$. So

$$\int_a^u f + \int_u^b f = \lim_{n\to\infty}\left(\int_a^u \phi_n + \int_u^b \phi_n\right) = \lim_{n\to\infty}\int_a^b \phi_n = \int_a^b f$$

by the additivity of integrals of step functions. ■

Proposition 11: The map $I : R[a,b] \to \mathbb{R}$, $I : f \mapsto \int_a^b f$ is linear.

Proof: The proof is similar to a combination of the last proof and the proof for the linearity of the integral for step functions, and so is left as an exercise. ■

Note: Not all functions are regulated.

Example: Consider $f : [0,1] \to \mathbb{R}$, with

$$f : x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

f is also denoted by $\mathbb{1}_{\mathbb{Q}}$ - the indicator of $\mathbb{Q}$. Let ϕ be any step function on $[a,b]$, with a compatible partition P, and let $\phi_1 = \phi\Big|_{(p_o,p_1)}$. Then $\|f-\phi\|_\infty \geq \left\|f\Big|_{(p_0,p_1)} - \phi_1\right\|_\infty = \max(|\phi_1|, |\phi_1 - 1|) \geq \frac{1}{2}$, so there cannot exist a sequence of step functions converging uniformly to f.

Proposition 12: Suppose that $f \in R[a,b]$, and that, $\forall x \in [a,b]$, $m \leq f(x) \leq M$ for some $m, M \in \mathbb{R}$. Then

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Proof: $f \in R[a,b]$, so, $\forall \varepsilon > 0$, $\exists \phi_\varepsilon \in S[a,b]$ such that $\|f - \phi_\varepsilon\|_\infty < \varepsilon$, so $\forall x \in [a,b]$, $f(x) - \varepsilon \leq \phi_\varepsilon(x) \leq f(x) + \varepsilon$, so $m - \varepsilon \leq \phi_\varepsilon(x) \leq M + \varepsilon$, now using the result proved for step functions

$$(m-\varepsilon)(b-a) \leq \int_a^b \phi_\varepsilon \leq (M-\varepsilon)(b-a)$$

Choose $\varepsilon = \frac{1}{n}$ for $n \in \mathbb{N}$, then $(\phi_n)_{n\geq 1}$ converges uniformly to f. Take the limit of the above with $\varepsilon = \frac{1}{n}$, then

$$m(b-a) \leq \int_a^b f \leq M(b-a) \quad ■$$

Exercise: Prove that, for any $f \in R[a,b]$,

$$\left|\int_a^b f\right| \leq \|f\|_\infty (b-a)$$

Example: $f(x) = x$ is uniformly continuous on $\mathbb{R}$, since $\left|f(x) - f(y)\right| = |x - y|$, so when $|x - y| < \delta = \varepsilon$, $\left|f(x) - f(y)\right| < \varepsilon$.

Proposition 13: Let $f_\alpha : [1, \infty) \to \mathbb{R}$, $f_\alpha : x \mapsto x^\alpha$ for some $\alpha \geq 0$. Then f_α is uniformly continuous for any $\alpha \geq 0$ but not uniformly continuous for $\alpha > 1$.

Proof: For $\alpha > 1$, $f_\alpha(x+\delta) - f_\alpha(x) = (x+\delta)^\alpha - x^\alpha = x^\alpha((1+\frac{\delta}{x})^\alpha - 1) \geq x^\alpha(1+\alpha\frac{\delta}{x} - 1) = \alpha\delta x^{\alpha-1} \to \infty$ as $x \to \infty$, which contradicts the condition for uniform continuity.

For $\alpha < 1,\ \alpha \neq 0$, $0 \leq f_\alpha(x+\delta) - f_\alpha(x) = x^\alpha((1+\frac{\delta}{x})^\alpha - 1) \leq x^\alpha(1 + \frac{\alpha\delta}{x} - 1)$, $(\alpha - 1) < 0$, so $x^\alpha \to 0$ as $x \to \infty$. For any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2\alpha\delta}$, then x^α is uniform continuous.

The case when $\alpha = 0$ is trivial and is left as an exercise. ■

Proposition 14: If $f : [a, b] \to \mathbb{R}$ is weakly increasing, then $f \in R[a, b]$.

Proof: For $f(b) > f(a)$, let $d = \frac{f(b)-f(a)}{n}$. Let P be a partition with $n + 1$ elements such that $p_0 = a$, $p_j = \sup_{x\in[a,b]}\{x : f(x) \leq f(a) + dj\}$, and $p_{n+1} = b$. By construction, since $p_j > p_{j-1}$, then $\forall x \in (p_{j-1}, p_j)$, $f(a) + d(j-1) \leq f(x) \leq f(a) + dj$. Let $\varphi_n \in S[a, b]$ such that $\varphi_n\Big|_{(p_{j-1},p_j)} = f(a) + dj$ and $\varphi_n(p_j) = f(p_j)$. Now, let $x \in (p_{j-1}, p_j)$. Then, using the previous inequality, $-d \leq f(x) - \varphi_n(x) \leq 0$, so $\|f - \varphi_n\|_\infty \leq d = \frac{f(b)-f(a)}{n} \to 0$ as $n \to \infty$, so f is regulated. ■

Exercise: Prove that if $f : [a, b] \to \mathbb{R}$ is weakly decreasing, then f is regulated. This proves that all monotonic functions on bounded intervals are regulated.

Example: Let $\mathbb{Q} \cap [0,\ 1] = \{\frac{0}{1},\ \frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{5},\ \ldots\} = \{q_1,\ q_2,\ q_3,\ \ldots\}$, and let

$$\mathbb{1}(x > q_n) = \begin{cases} 1 & x > q_n \\ 0 & x \leq q_n \end{cases}$$

Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f : x \mapsto \sum_{n=1}^{\infty} 2^{-n}\mathbb{1}(x > q_n)$$

which exists $\forall\ x \in [0, 1]$. Then f is non-decreasing, and is regulated, but jumps an infinitely many times, so is impossible to draw.

3 The Indefinite Integral & Fundamental Theorem of Calculus

3.1 The Indefinite Integral

Definition 6: Let $f \in R[a,b]$. Define $F : [a,b] \to \mathbb{R}$ by

$$F : x \mapsto \int_a^x f$$

F is called the *indefinite integral* of f.

Lemma: $R[a,b] \subset B[a,b]$

Proof: $f \in R[a,b]$, so $\exists \phi \in S[a,b]$ such that $\|\phi - f\|_\infty \leq 1$, so $\forall\, x \in [a,b]$, $\phi(x) - 1 \leq f(x) \leq \phi(x) + 1$. Since step functions are bounded, $\phi \in B[a,b]$, so $f \in B[a,b]$. ■

Proposition 15: F is continuous on $[a,b]$.

Proof: Let $x,\ y \in [a,b]$. Then

$$\big|F(x) - F(y)\big| = \left|\int_a^x f - \int_a^y f\right| = \left|\int_y^x f\right| \leq \|f\|_\infty |x-y|\,,$$

so by the sequential definition of continuity, f is continuous.

Note: Let $f : [a,b] \to \mathbb{R}$. Then, if $\exists\, L > 0$ such that, $\forall\, x, y \in [a,b]$, $\big|f(x) - f(y)\big| \leq L|x-y|$. Then f is called *Lipschitz continuous* or *Lipschitz*. Lipschitz continuity implies continuity, but the reverse is not generally true.

Example: $f(x) = e^{-x}$ is Lipschitz on $[0,1]$, but $g(x) = \sqrt{x}$ is not Lipschitz on $[0,1]$.

3.2 The Fundamental Theorem of Calculus

Theorem 16 - FTC1: Let $f \in R[a,b]$ such that f is continuous at $c \in (a,b)$. Then $F(x) = \int_a^x f$ is differentiable at c and $F'(c) = f(c)$.

Proof: f is continuous at c, so, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, $\forall x \in (c-\delta, c+\delta)$, $\big|f(x) - f(c)\big| < \varepsilon$, so $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. Choose $0 < h < \delta$. Then, by integrating the previous statement

$$h(f(c)-\varepsilon) < \int_c^{c+h} f < h(f(c)+\varepsilon), \text{ so } h(f(c)-\varepsilon) < \int_a^{c+h} f - \int_a^c f = F(c+h) - F(c) < h(f(c)+\varepsilon)$$

So, for any $h \in (0,\ \delta)$, $h(f(c)-\varepsilon) < F(c+h) - F(c) < h(f(c)+\varepsilon)$, so

$$-\varepsilon < \frac{F(c+h) - F(c)}{h} - f(c) < \varepsilon$$

So F is differentiable at c, and $F'(c) = f(c)$. The proof for the case where $-\delta < h < 0$ is similar and so is left as an exercise. ■

Corollary 17: If $f \in C[a,b]$, then f is differentiable on (a,b) and $F' = f$.

Proof: f is continuous, so it is regulated. Applying FTC1 completes the proof. ■

Theorem 18 - FTC2: Let $f : [a,b] \to \mathbb{R}$ be continuous, and let $g : [a,b] \to \mathbb{R}$ be differentiable, with $g'(x) = f(x)\ \forall x \in [a,b]$. Then

$$\int_a^b f = g(b) - g(a)$$

This is the *Newton-Leibniz Formula*, and g is called the *antiderivative* of f. This is sometimes written

$$\int_a^b g' = g(b) - g(a)$$

Proof: Let $\delta : [a,b] \to \mathbb{R}$, with $\delta(x) = g(x) - \int_a^x f$. Then δ is continuous on $[a,b]$. it is also differentiable on (a,b) by FTC1. By the MVT, $\delta(b) - \delta(a) = \delta(\xi)(b-a)$ for some $\xi \in (a,b)$. But $\delta'(\xi) = g'(\xi) - f(\xi)$ by FTC1, so $\delta'(\xi) = f(\xi) - f(\xi) = 0$, so $\delta(a) = \delta(b)$, so $g(a) - \int_a^a f = g(b) - \int_a^b f$, so $\int_a^b f = g(b) - g(a)$. ■

4 Practical Methods of Integration

4.1 Integration by Parts

Proposition 20: If $f, g \in R[a,b]$, then $f \cdot g \in R[a,b]$.

Proof: $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Then $\exists (\varphi_n), (\psi_n) \in S[a,b]$ such that $(\varphi_n) \to f, (\psi_n) \to g$ uniformly. $0 \leq \|f \cdot g - \varphi_n \psi_n\|_\infty = \|f \cdot g - \varphi_n g + \varphi_n g - \varphi_n \psi_n\|_\infty \leq \big\|(f-\varphi_n)g\big\|_\infty + \big\|\varphi_n(g-\psi_n)\big\|_\infty$.

Since $\sup_{x\in[a,b]}\big|a(x)\cdot b(x)\big| \leq \sup_{x\in[a,b]}\big|a(x)\big| \cdot \sup_{x\in[a,b]}\big|b(x)\big|$, this is less than or equal to $\|g\|_\infty \|f-\varphi_n\|_\infty + \|\varphi_n\|_\infty \|g-\psi_n\|_\infty$. Now, $\|g\|_\infty$ is regulated, and so is bounded. $\|f-\varphi_n\|_\infty, \|g-\psi_n\|_\infty \to 0$ as $n \to \infty$, and $\|\varphi_n\|_\infty = \|\varphi_n - f + f\|_\infty \leq \|\varphi_n - f\|_\infty + \|f\|_\infty$. Also, $\|\varphi_n - f\|_\infty \to 0$ as $n \to \infty$, and $\|f\|_\infty$ is regulated, and so is bounded. Taken together, this means that $\|f\cdot g - \varphi_n\psi_n\|_\infty \to 0$ as $n \to \infty$, so $(\varphi_n\psi_n)$ converges uniformly to $f \cdot g$, so $f \cdot g$ is regulated. ■

Corollary 19 - Integration by Parts: If F and G are differentiable such that F' and G' are continuous, hence regulated, then

$$\int_a^b F'G = F(b)G(b) - F(a)G(a) - \int_a^b FG'$$

Proof: $(FG)' = FG' + F'G$. Then

$$\int_a^b (FG)' = \int_a^b FG' + F'G = \int_a^b FG' + \int_a^b F'G$$

By FTC2

$$F(b)G(b) - F(a)G(a) = \int_a^b FG' + F'G = \int_a^b FG' + \int_a^b F'G$$

So

$$\int_a^b F'G = F(b)G(b) - F(a)G(a) - \int_a^b FG' \quad ■$$

Note: $F(b)G(b) - F(a)G(a)$ can be denoted by $FG|_a^b$.

Example: The *Gamma Function* $\Gamma : (0,\infty) \to \mathbb{R}$ is given by

$$\Gamma : x \mapsto \int_0^\infty t^{x-1}e^{-t}\,dt := \lim_{\varepsilon\downarrow 0, L\to\infty} \int_\varepsilon^L t^{x-1}e^{-t}\,dt$$

Then

$$\begin{aligned}\Gamma(x) &= \lim_{\varepsilon\downarrow 0, L\to\infty} \int_\varepsilon^L \left(\frac{1}{x}t^x\right)' e^{-t}\,dt \\ &= \frac{1}{x} \lim_{\varepsilon\downarrow 0, L\to\infty} (t^x e^{-t})|_\varepsilon^L - \int_\varepsilon^L t^x 1(-e^{-t})\,dt \\ &= \frac{1}{x}\int_0^\infty t^x e^{-t}\,dt \\ &= \frac{1}{x}\Gamma(x+1)\end{aligned}$$

So $\Gamma(x+1) = x\Gamma(x)$, and $\Gamma(1) = 1$, so $\Gamma(n+1) = n!$

4.2 Integration by Substitution

Corollary 21 - Integration by Substitution: Let $f \in C[a,b]$, and let $g : [c,d] \to [a,b]$ be differentiable with $\operatorname{Im} g \subset [a,b]$. Then

$$\int_c^d f(g(x)) \cdot g'(x)\ dx = \int_{g(c)}^{g(d)} f(t)\ dt$$

Proof: Let $F(x) = \int_a^x f$. Since f is continuous, F is differentiable and $F'(x) = f(x)$ by FTC1, then by FTC2

$$\int_{g(c)}^{g(d)} f(t)\ dt = F(g(d)) - F(g(c))$$

Also, $\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$ which is the initial integrand. Now, by FTC2, $F(g(x))|_c^d = F(g(d)) - F(g(c))$, so

$$\int_c^d f(g(x)) \cdot g'(x)\ dx = \int_{g(c)}^{g(d)} f(t)\ dt \quad \blacksquare$$

Note: g does not need to be injective or surjective.

Example: Let $g(x) = x^2$. Then

$$\int_{-a}^{a} f(x^2)(x^2)'\ dx = F(x^2)|_{-a}^{a}$$

Example: Let

$$G(a,j) := \int_{-\infty}^{\infty} e^{-ax^2+2jx}\ dx = \lim_{L_1\to\infty, L_2\to\infty} \int_{-L_2}^{L_1} e^{-ax^2+2jx}\ dx$$

for some $a > 0$, $j \in \mathbb{R}$. Then

$$\begin{aligned} G(a,j) &= \int_{-\infty}^{\infty} e^{-ax^2+2jx}\ dx \\ &= \int_{-\infty}^{\infty} e^{-a(x^2+\frac{2j}{a}x)}\ dx \\ &= \int_{-\infty}^{\infty} e^{-a((x-\frac{j}{a})^2-\frac{j^2}{a^2})}\ dx \\ &= e^{\frac{j^2}{a}} \int_{-\infty}^{\infty} e^{-a(x-\frac{j}{a})^2}\ dx \end{aligned}$$

Now, substituting $g(x) = x - \frac{j}{a}$,

$$e^{\frac{j^2}{a}} \int_{-\infty}^{\infty} e^{-a(x-\frac{j}{a})^2}\ dx = e^{\frac{j^2}{a}} \int_{-\infty}^{\infty} e^{-at^2}\ dt = e^{\frac{j^2}{a}} \int_{-\infty}^{\infty} e^{-(\sqrt{a}t)^2}\ dt$$

Now, substituting $h(t) = \sqrt{a}t$,

$$e^{\frac{j^2}{a}} \int_{-\infty}^{\infty} e^{-(\sqrt{a}t)^2}\ dt = e^{\frac{j^2}{a}} \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2}\ dy = e^{\frac{j^2}{a}} \sqrt{\frac{\pi}{a}}$$

So

$$G(a,\ j) = e^{\frac{j^2}{a}} \sqrt{\frac{\pi}{a}}$$

4.3 Constructing Convergent Sequences of Step Functions

Theorem 22: Let $f : [a, b] \to \mathbb{R}$ be regulated. Then

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{b-a}{h} f(a_j) = \int_a^b f(x)\,dx, \text{ where } a_j = a + \frac{b-a}{h} j$$

Proof: First, consider $\phi \in S[a, b]$. Let P be a partition compatible with ϕ, with $|P| = k$. Then

$$0 \leq \left| \int_a^b \phi - \sum_{j=1}^{n} \frac{b-a}{h} \phi(a_j) \right| \leq k \frac{b-a}{n} 2\|\phi\|_\infty \to 0 \text{ as } n \to \infty$$

since errors only occur and contribute to the difference at jumps on the step function, of which there are k, and the maximum error this can be is twice the maximum value of the step function across the whole section. Then

$$\int_a^b \phi = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{b-a}{h} \phi(a_j)$$

Now, for $f \in R[a, b]$, let

$$R_f^{(n)} = \sum_{j=1}^{n} \frac{b-a}{h} f(a_j), \text{ where } a_j = a + \frac{b-a}{h} j$$

Since f is regulated, then ,$\forall \varepsilon > 0$, $\exists \phi \in S[a, b]$ such that $\|f - \phi\|_\infty < \frac{\varepsilon}{3}$. Then

$$\begin{aligned} \left| R_f^{(n)} - \int_a^b f \right| &= \left| R_f^{(n)} - R_\phi^{(n)} + R_\phi^{(n)} - \int_a^b \phi + \int_a^b \phi - \int_a^b f \right| \\ &\leq \left| R_f^{(n)} - R_\phi^{(n)} \right| + \left| R_\phi^{(n)} - \int_a^b \phi \right| + \left| \int_a^b \phi - f \right| \end{aligned}$$

Now,

$$\left| R_f^{(n)} - R_\phi^{(n)} \right| \leq \|\phi - f\|_\infty (b-a) < \frac{\varepsilon}{3}(b-a)$$

and

$$\left| R_\phi^{(n)} - \int_a^b \phi \right| < \frac{\varepsilon}{3}(b-a) \text{ eventually in } n$$

and

$$\left| \int_a^b \phi - f \right| = \left| \sum_{j=1}^{n} (f(a_j) - \phi(a_j)) \frac{b-a}{n} \right| \leq \sum_{j=1}^{n} \left| f(a_j) - \phi(a_j) \right| \frac{b-a}{n} < \frac{\varepsilon}{3}(b-a)$$

since $\left| f(a_j) - \phi(a_j) \right| < \frac{\varepsilon}{3}$,

so

$$\left| R_f^{(n)} - \int_a^b f \right| < \varepsilon(b-a) \text{ eventually in } n$$

so

$$\lim_{n\to\infty} R_f^{(n)} = \int_a^b f \quad \blacksquare$$

5 The Characterisation of Regulated Functions

5.1 Characterisation & Proof

Proposition 23: A function $f \in R[a,b]$ iff, $\forall x \in (a,b)$, $f(x\pm)$ both exist, and both $f(a+)$ and $f(b-)$ exist, where $f(x+) = \lim_{y \downarrow x} f(y)$ (the right limit), and $f(x-) = \lim_{y \uparrow x} f(y)$ (the left limit). Alternatively, this can be stated as "all one-sided limits exist".

Note: This characterisation only requires the existence of these limits, not equality, $f(x+) \neq f(x-)$ is acceptable so long as they both exist.

Proof $\implies$: f is regulated, so, $\forall \varepsilon > 0$, $\exists \phi \in S[a,b]$ such that $\|f - \phi\|_\infty < \varepsilon$. Let $x \in (a,b)$. There are now two cases, ϕ is continuous at x, or ϕ is discontinuous at x. In either case, $\exists \delta > 0$ such that $\phi\Big|_{(x,x+\delta)} = \phi_0$, a constant. Let $(x_n)_{n\geq 1} \subset \mathbb{R}$ be any sequence such that, $\forall n \in \mathbb{N}$, $x_n > x$, and $\lim_{n\to\infty} x_n = x$. Eventually $x_n \in (x, x+\delta)$. Then, eventually in n and m, $\left|f(x_n) - f(x_m)\right| = \left|f(x_n) - \phi_0 + \phi_0 - f(x_m)\right| \leq \left|f(x_n) - \phi_0\right| + \left|f(x_m) - \phi_0\right| < 2\varepsilon$, since $\left|f(x_n) - \phi_0\right|, \left|f(x_m) - \phi_0\right| < \varepsilon$, so $f(x_n)_{n\geq 1}$ is Cauchy, and so converges. If y_n is any other sequence with $y_n \downarrow x$ as $n \to \infty$, then $(f(y_n))_{n\geq 1}$ converges to the same limit. The proof of this fact is routine after considering $(x_n, y_n)_{n\geq 1}$, and so is left as an exercise. It has been shown that for any $(x_n)_{n\geq 1} \subset \mathbb{R}$ such that $x_n \downarrow x$ as $n \to \infty$, $\lim_{n\to\infty} f(x_n) = f(x+)$ exists and does not depend on x_n. A similar argument can be used to prove this result for the left limit, and not much modification is required for $x = a, b$, so these proofs are left as exercises.

Proof $\impliedby$: All one-sided limits exist. Now, fix any $\varepsilon > 0$. Let

$$A_\varepsilon = \{\gamma \in [a,b] : \exists \psi \in S[a,b] \text{ such that } \left\| (f - \psi)\Big|_{[a,\gamma]} \right\|_\infty < \varepsilon\}$$

The proof shall follow from proving three statements:

(i) $A_\varepsilon \neq \varnothing$: $f(a+)$ exists, so, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, $\forall x \in (a, a+\delta)$, $\left|f(x) - f(a+)\right| < \varepsilon$. Take $\gamma = \frac{\delta}{2}$. Then define

$$\phi(x) = \begin{cases} f(a) & x = a \\ f(a+) & x \in (a, a + \frac{\delta}{2}) \end{cases}$$

It is clear that $\phi(x) \in S[a, a + \frac{\delta}{2}]$. Since $\left|f(x) - f(a+)\right| < \varepsilon$ and $\left|f(a) - \phi(a)\right| = 0$, $\left\| (f - \phi)\Big|_{[a, a+\frac{\delta}{2}]} \right\|_\infty < \varepsilon$, so $\gamma = \frac{\delta}{2} \in A_\varepsilon$, so $A_\varepsilon \neq \varnothing$.

(ii) $\sup A_\varepsilon = b$: Assume that $c \coloneqq \sup A_\varepsilon < b$. Then, $\forall \delta > 0$, $(c - \delta) \in A_\varepsilon$, so $\exists \phi \in S[a, c-\delta]$ such that $\left\| (f - \phi)\Big|_{[a, c-\delta]} \right\|_\infty < \varepsilon$. But f has limits $f(c\pm)$ at c, so $\forall \varepsilon > 0$, $\exists \delta_1, \delta_2$ such that, $\forall x \in (c, c+\delta_2)$, $\left|f(x) - f(c+)\right| < \varepsilon$ and, $\forall x \in (c - \delta_1, c)$, $\left|f(x) - f(c-)\right| < \varepsilon$. Choose $\delta < \delta_1$. Now, let $\psi \in S[a,b]$ be given by

$$\psi(x) = \begin{cases} f(c-) & x < c \\ f(c) & x = c \\ f(c+) & x > c \end{cases}$$

and let $\tilde{\phi} \in S[a,b]$ be given by

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & x \in [a, c-\delta] \\ \psi(x) & x \in (c - \delta, c + \frac{\delta_2}{2}] \end{cases}$$

Then $(c + \frac{\delta_2}{2}) \in A_\varepsilon$, with $\frac{\delta_2}{2} > 0$, which contradicts the fact that $c = \sup A_\varepsilon$, so $b = c$.

(iii) $b \in A_\varepsilon$: It has been shown that, $\forall \delta > 0$, $\exists \phi \in S[a, b-\delta]$ such that $\left\| (f-\phi)\big|_{[a,b-\delta]} \right\|_\infty < \varepsilon$. But $f(b-)$ exists, so $\exists \delta_1 > 0$ such that, $\forall x \in (b-\delta_1, b)$, $\left|f(x) - f(b-)\right| < \varepsilon$. Choose $\delta < \delta_1$. Now, let $\psi \in S[a,b]$ be given by

$$\psi(x) = \begin{cases} f(b) & x = b \\ f(b-) & x < b \end{cases}$$

and let $\tilde{\phi} \in S[a,b]$ be given by

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & x \in [a, b-\delta] \\ \psi(x) & x \in (b-\delta, b] \end{cases}$$

Now, by definition, $\left\| f - \tilde{\phi} \right\|_\infty < \varepsilon$, so $b \in A_\varepsilon$.

So $\gamma = b$, so $\exists \varphi \in S[a,b]$ such that $\|f - \varphi\|_\infty < \varepsilon$, so $f \in R[a,b]$. ■

Exercise: Prove that if $f \in R[a,b]$, then the set of discontinuities of f is countable. *Hint:* Check that, $\forall \varepsilon > 0$, the set of jumps of size ε or greater is countable.

6 Improper & Riemann Integrals

6.1 Improper Integrals

Motivation: It is possible to generalise the regulated integral to unbounded functions and/or unbounded domains.

Example:

$$\int_0^1 x^{-\frac{1}{2}}\, dx = 2$$

because, although $x^{-\frac{1}{2}}$ is not regulated on $(0, 1]$, $\forall \varepsilon > 0$, $x^{-\frac{1}{2}} \in R[\varepsilon, 1]$. Then

$$\int_\varepsilon^1 x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}}|_\varepsilon^1 = 2 - 2\varepsilon^{\frac{1}{2}} \to 2 \text{ as } \varepsilon \to 0$$

Example:

$$\int_1^\infty \frac{1}{x^2}\, dx = 1$$

because, $\forall L > 1$, $\frac{1}{x^2} \in R[1, L]$. Then

$$\int_1^L \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_1^L = 1 - \frac{1}{L} \to 1 \text{ as } L \to \infty$$

Example:

$$\int_1^0 \frac{1}{x} dx = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \frac{1}{x}\, dx = \lim_{\varepsilon \downarrow 0} \log x|_\varepsilon^1 = \lim_{\varepsilon \downarrow 0} \log \frac{1}{\varepsilon}$$

which does not exist, therefore the integral is not defined.

Definition 7.1: If, for any $\varepsilon > 0$, $f : (a, b] \to \mathbb{R}$ is regulated on $[a + \varepsilon, b]$ and

$$l := \lim_{\varepsilon \downarrow 0} \int_{a+\varepsilon}^b f$$

exists, then $\int_a^b f$ *converges* and is equal to l. If the limit does not exist, then $\int_a^b f$ *diverges*.

Definition 7.2: If, $\forall L > 0$, $f \in R[a, L]$ and

$$l := \lim_{L \to \infty} \int_a^L f$$

exists, then $\int_a^\infty f$ *converges* and is equal to l. If the limit does not exist, then $\int_a^\infty f$ *diverges*.

Note: Let $f \in R[-L_1, L_2]$ for some $L_1, L_2 > 0$. Then $\int_{-\infty}^{\infty} f$ converges if, $\forall c \in \mathbb{R}$, $\int_{-\infty}^{c} f$ and $\int_{c}^{\infty} f$ converge. In this case, $\int_{-\infty}^{c} f + \int_{c}^{\infty} f = \int_{-\infty}^{\infty} f$. As an exercise, show that if $\int_{-\infty}^{c} f$ and $\int_{c}^{\infty} f$ exist, then $\int_{-\infty}^{c} f + \int_{c}^{\infty} f$ does not depend on the choice of c.

Example: Let $f(x) = x \in R[-L, L]$ $\forall L > 0$. Then

$$\int_{-L}^{L} x\ dx = 0, \text{ so } \lim_{L\to\infty} \int_{-L}^{L} x\ dx = 0$$

However, take any $c \in \mathbb{R}$. Then

$$\lim_{L\to\infty} \int_{c}^{L} x\ dx = \infty, \text{ and } \lim_{L\to\infty} \int_{-L}^{c} x\ dx = -\infty$$

so do not exist. Then

$$\int_{-\infty}^{\infty} x\ dx$$

does not exist, since the limits must exist independently.

Notes: Suppose $f : (a, b) \to \mathbb{R}$, and, for any $\varepsilon_1, \varepsilon_2 > 0$, $f \in R[a + \varepsilon_1, b - \varepsilon_2]$. Then $\int_a^b f$ exists if, for any $c \in (a, b)$, $\int_a^c f$ and $\int_c^b f$ both exist (converge). Then $\int_a^b f$ converges, and $\int_a^b f = \int_a^c f + \int_c^b f$.

Suppose f is defined on $[a, b) \cup (b,\ c]$. Then $\int_a^c f$ exists (converges) if $\int_a^b f$ and $\int_b^c f$ both exist (independently), with $\int_a^c f = \int_a^b f + \int_b^c f$.

Example: $f(x) = \frac{1}{\sqrt{|x|}}$ for $x \in [-1, 0) \cup (0, 1]$.

Note: Let $f : (a, b) \to \mathbb{R}$ such that, for any $\varepsilon_1, \varepsilon_2 > 0$, $f \in [a + \varepsilon_1, b - \varepsilon_2]$. Then $\exists c \in (a, b)$ such that

$$l_1(c) = \lim_{\varepsilon_1 \downarrow 0} \int_{a+\varepsilon_1}^{c} f \text{ and } l_2(c) = \lim_{\varepsilon_2 \downarrow 0} \int_{c}^{b-\varepsilon_2} f$$

exist iff, $\forall c' \in (a, b)$, $l_1(c')$ and $l_2(c')$ exist. The first formulation is often the more convenient in practice however. If this formulation is true of f, then $\int_a^b f$ converges, and $\int_a^b f = l_1(c) + l_2(c)$. As an exercise, check that $l_1(c) + l_2(c) - l_1(c') - l_2(c') = 0$ to demonstrate that this integral is independent of the choice of c.

Definition 7.3: Let $f \in R[-L_1, L_2]$ for some $L_1, L_2 > 0$. Then

$$\int_{-\infty}^{\infty} f := \lim_{L_1\to\infty} \int_{c}^{L_1} f + \lim_{L_2\to\infty} \int_{-L_2}^{c} f$$

Definition 7.4: If $f : [a, b) \cup (b,\ c] \to \mathbb{R}$, then

$$\int_{a}^{c} f := \lim_{\varepsilon_1 \downarrow 0} \int_{a}^{b-\varepsilon_1} f + \lim_{\varepsilon_2 \downarrow 0} \int_{b+\varepsilon_2}^{c} f$$

Note: All integrals in the limit must make sense as regulated or improper integrals.

Note: It is possible to define the integral of functions with domains such as

$$f : \bigcup_{j\in\mathbb{Z}} (j,\ j+1) \to \mathbb{R}$$

Situations like this occur when studying gamma functions.

6.2 The Riemann Integral

Definition: Let $f : [a, b] \to \mathbb{R}$. Then

$$U_f := \inf_{\phi \in S[a,b]} \left\{ \int_a^b \phi : \phi \geq f \right\} \text{ ("upper sum")}$$

$$L_f := \sup_{\phi \in S[a,b]} \left\{ \int_a^b \phi : \phi \leq f \right\} \text{ ("lower sum")}$$

Definition: $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* if $U_f = L_f$. Then

$$R\!\!\!\!\int_a^b f := U_f = L_f$$

Proposition: If $f \in R[a, b]$, then it is Riemann integrable, and

$$R\!\!\!\!\int_a^b f = \int_a^b f$$

Proof: $f \in R[a, b]$, so, $\forall \varepsilon > 0$, $\exists \phi \in S[a, b]$ such that, $\forall x \in [a, b]$, $\phi(x) - \varepsilon \leq f(x) \leq \phi(x) + \varepsilon$. Since $\phi(x) - \varepsilon, \phi(x) + \varepsilon \in S[a, b]$,

$$\int_a^b \phi - \varepsilon(b - a) \leq L_f \leq U_f \leq \int_a^b \phi + \varepsilon(b - a)$$

Taking $\varepsilon \downarrow 0$,

$$L_f = U_f = \int_a^b f = R\!\!\!\!\int_a^b f \quad \blacksquare$$

Note: There are functions which are Riemann integrable but not regulated.

Example: Let $f : [0,\ 1] \to \mathbb{R}$ be given by

$$f : x \mapsto \begin{cases} 1 & x = 2^{-n} \text{ for } n = 0,\ 1,\ 2,\ \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then f is not regulated since, $\forall p > 0$, $\exists x \in (0, p)$ such that $f(x) = 1$ and $\exists y \in (0, p)$ such that $f(y) = 0$, so a step function can get no closer than a half to both "sections" of the function on any interval, so $\int_a^b f$ is not defined.

However, let

$$\phi_N = \begin{cases} 1 & x \leq 2^{-N} \\ f(x) & x > 2^{-N} \end{cases}$$

Now, $\phi_N \geq f \geq 0$, so

$$0 \leq L_f \leq U_f \leq \int_0^1 \phi_N = \lim_{N \to \infty} \int_{2^{-N}}^1 \phi_N = 0$$

Then $L_f = U_f = 0$, so

$$R\!\!\!\!\int_0^1 f = 0$$

so the Riemann integral of f is defined.

Note: There are functions which are not Riemann integrable.

Example: Let

$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then $U_{\mathbb{1}_{\mathbb{Q}}} = 1 \neq 0 = L_{\mathbb{1}_{\mathbb{Q}}}$, so $\mathbb{1}_{\mathbb{Q}}$ is not Riemann integrable. However, it is *Lebesgue integrable*, with

$$\mathcal{L}\!\!\!\int_a^b \mathbb{1}_{\mathbb{Q}} = 0$$

7 Uniform & Pointwise Convergence

7.1 Definitions, Properties and Examples

Example: Let $(f_n)_{n\geq 1}$ be a sequence of functions on [-1,1], with

$$f_n : x \mapsto x^{-\frac{1}{2^n-1}} \text{ for } n = 1, 2, \ldots$$

Fix x and consider

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}, \text{ so } \lim_{n\to\infty} f_n(x) = \operatorname{sign} x.$$

So the limit of $(f_n(x))_{n\geq 1}$ as $n \to \infty$ exists $\forall x \in [-1, 1]$, and f_n is continuous $\forall n \in \mathbb{N}$, but the limit is not continuous at 0.

$$\lim_{n\to\infty} \lim_{x\to 0\pm} f_n(x) = 0 \neq \pm 1 = \lim_{x\to 0\pm} \operatorname{sign} x = \lim_{x\to 0\pm} \lim_{n\to\infty} f_n(x)$$

This function has non-commutative limits and a sequence of continuous functions converges to a discontinuous function, both of which are undesirable properties of *pointwise convergence.*

Definition 8: Let $(f_n)_{n\geq 1}$ be a sequence of functions, and let $A \subset \mathbb{R}$. Then $f_n \to f$ *pointwise* as $n \to \infty$ if, $\forall x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Note: Pointwise convergence is very "loose", it does not preserve properties of the f_ns such as continuity (as in the case of $x^{-\frac{1}{2^n-1}} \to \operatorname{sign} x$). Pointwise convergence can also be very non-uniform (the same example converges pointwise very slowly near 0).

Note: Let f and $(f_n)_{n\geq 1}$ be functions on $A \subset \mathbb{R}$. If $(f_n) \to f$ pointwise as $n \to \infty$, then f can fail to be continuous, even if all f_ns are. Alternatively, $\exists x_0 \in A$ such that

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) \neq \lim_{x\to x_0} \lim_{n\to\infty} f_n(x)$$

Example (The Witch's Hat): Let $f_n : [0, 1] \to \mathbb{R}$, with $f_n(0) = 0$ and

$$f_n(x) = \begin{cases} 2n^2 x & x \in [0, \frac{1}{2n}] \\ n - 2n^2(x - \frac{1}{2n}) & x \in (\frac{1}{2n}, \frac{1}{n}] \\ 0 & x > \frac{1}{n} \end{cases}$$

Then $f_n \in C[0, 1]$. Fix $x > 0$, then $\lim_{n\to\infty} f_n(x) = 0$. Now, as $n \to \infty$, $f_n \to f \equiv 0$ on $[0, 1]$, which is continuous. However, $\int_0^1 f = 0$, but $\int_0^1 f_n = n \cdot \frac{1}{2n} = \frac{1}{2}$, so

$$\frac{1}{2} = \lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 \lim_{n\to\infty} f_n = 0$$

It is clear from examples such as this that pointwise convergence must be strengthened.

Definition 9: Let $f, f_n : A \subset \mathbb{R} \to \mathbb{R}$ for $n = 1, 2, \ldots$, then $(f_n)_{n\geq 1} \to f$ *converges uniformly* as $n \to \infty$ if, $\forall \varepsilon > 0$, $\exists N_\varepsilon$ such that, $\forall n > N_\varepsilon, \left|f_n(x) - f(x)\right| < \varepsilon \ \forall x \in A$, or, equivalently, if $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$.

Note: For the remainder of this section, assume, unless specified otherwise, that $f,\ f_n : A \subset \mathbb{R} \to \mathbb{R}$ for $n = 1,\ 2,\ \ldots$.

Remark: Uniform convergence $\implies$ pointwise convergence, but pointwise convergence $\not\Longrightarrow$ uniform convergence.

Definition 9.1: A sequence $(f_n)_{n\geq 1}$ (of functions on $A \subset \mathbb{R}$) is called *uniformly Cauchy* if, $\forall \varepsilon > 0$, $\exists N_\varepsilon$ such that, $\forall n, m > N_\varepsilon, \left|f_n(x) - f_m(x)\right| < \varepsilon\ \forall x \in A$.

Note: For a fixed $x \in A$, if $(f_n(x))_{n\geq 1}$ is Cauchy, then it is pointwise convergent (to a common limit f).

Lemma: A uniformly Cauchy sequence converges uniformly.

Proof: Let $(f_n)_{n\geq 1}, (f_m)_{m\geq 1}$ be uniformly Cauchy sequences converging to f. Then, $\forall \varepsilon > 0$, $\exists N_\varepsilon$ such that, $\forall n, m > N_\varepsilon$, $f_m(x) - \frac{\varepsilon}{2} \leq f_n(x) \leq f_m(x) + \frac{\varepsilon}{2}\ \forall x \in A$. Take $m \to \infty$, then $f_m \to f$, so $f(x) - \frac{\varepsilon}{2} \leq f_n(x) \leq f(x) + \frac{\varepsilon}{2}\ \forall x \in A$, so $\left|f_n(x) - f(x)\right| < \varepsilon\ \forall x \in A$, so $(f_n)_{n\geq 1}$ converges uniformly. ∎

Theorem 24: Let $(f_n)_{n\geq 1} \subset R[a, b]$. Assume that $f_n \to f$ uniformly. Then

(i) $f \in\ R[a, b]$

(ii) $\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f$

Proof:

(i) $f_n \in R[a, b]$, so, $\forall \varepsilon > 0$, $\exists \phi_n \in S[a, b]$ such that $\|f_n - \phi_n\|_\infty < \frac{\varepsilon}{2}$. Now, $\|f - \phi_n\|_\infty = \|f - f_n + f_n - \phi_n\|_\infty \leq \|f - f_n\|_\infty + \|f_n - \phi_n\|_\infty$. Assume that n is large enough such that $\|f - f_n\|_\infty < \frac{\varepsilon}{2}$, which is guaranteed since $f_n \to f$, so $\|f - \phi_n\|_\infty < \varepsilon$, so $f \in R[a, b]$.

(ii) It was shown in (i) that f is regulated, so $\int_a^b f$ is well defined and

$$0 \leq \left|\int_a^b f - \int_a^b f_n\right| \leq \|f - f_n\|_\infty (b - a)$$

$\|f - f_n\|_\infty \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f \quad ∎$$

Theorem 25: Suppose that $(f_n)_{n\geq 1} \subset C[a, b]$, and $f_n \to f$ uniformly as $n \to \infty$. Then $f \in C[a, b]$. This theorem also holds for some arbitrary $A \subset \mathbb{R}$ in place of $[a, b]$, and shall be proven for this more general case.

Proof: $f_n \to f$ uniformly, so $\forall \varepsilon > 0$, $\exists N_\varepsilon$ such that, $\forall n > N_\varepsilon, \left|f_n(x) - f(x)\right| < \varepsilon\ \forall x \in A$, so f_n is continuous $\forall n \in \mathbb{N}$. Then $\exists \delta'_\varepsilon$ such that, $\forall x \in (x_0 - \delta'_\varepsilon,\ x_0 + \delta'_\varepsilon) \cap A, \left|f_n(x) - f_n(x_0)\right| < \frac{\varepsilon}{3}$ $\forall x \in (x_0 - \delta'_\varepsilon, x_0 + \delta'_\varepsilon) \cap A$. Set $\delta_\varepsilon = \delta'_\varepsilon$. Then, $\forall x \in (x_0 - \delta'_\varepsilon,\ x_0 + \delta'_\varepsilon) \cap A$, $\left|f(x) - f(x_0)\right| = \left|f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)\right| \leq \left|f_n(x) - f_n(x_0)\right| + \left|f_n(x) - f_n(x_0)\right| + \left|f_n(x_0) - f(x_0)\right|$. $f_n \to f$ uniformly as $n \to \infty$, so the first and last of these terms are less than $\frac{\varepsilon}{3}$, and the middle term was shown to be less than $\frac{\varepsilon}{3}$ earlier, so $\left|f(x) - f(x_0)\right| < \frac{3\varepsilon}{3} = \varepsilon$, so f is continuous at x_0. ∎

Example (Cantor's Function or The Devil's Staircase): Let $(f_n)_{n\geq 1} \subset C[0,1]$ be such that $f_0 \equiv 1$ and

$$f_{n+1}(x) = \begin{cases} \frac{1}{2}f_n(3x) & x \in [0,\frac{1}{3}) \\ \frac{1}{2} & x \in [\frac{1}{3},\frac{2}{3}] \\ \frac{1}{2}+\frac{1}{2}f_n(3x-2) & x \in (\frac{2}{3},1] \end{cases}$$

Proposition: $\lim_{n\to\infty} f_n$ exists and is continuous. It is called *Cantor's function.*

Proof: Clearly $f_0 \in C[0,1]$. Now, suppose that $f_n \in C[0,1]$. Then f_{n+1} is clearly continuous on $(\frac{1}{3},\frac{2}{3})$. Now, f_n is continuous on $[0,\frac{1}{3})$ by assumption, so $\frac{1}{2}f_n(3\cdot)$ is continuous, so f_{n+1} is continuous on $[0,\frac{1}{3})$. Also, f_n is continuous on $(\frac{2}{3},1]$ by assumption, so $\frac{1}{2}+\frac{1}{2}f_n(3\cdot-2)$ is continuous, so f_{n+1} is continuous on $(\frac{2}{3},1]$. Now the only points remaining to investigate the continuity of are $\frac{1}{3}$ and $\frac{2}{3}$.

$f_{n+1}(1) = \frac{1}{2}+\frac{1}{2}f_n(1)$, which is 1 if $f_n(1) = 1$. $f_0(1) = 1$, so $f_n(1) = 1$ $\forall n \in \mathbb{N}$ by induction.

Concerning the continuity of $\frac{1}{3}$, $f_{n+1}(\frac{1}{3}) = \frac{1}{2}$, and $f_{n+1}(\frac{1}{3}+) = \frac{1}{2}$, so f_{n+1} is right-continuous at $\frac{1}{3}$, and $f_{n+1}(\frac{1}{3}-) = \frac{1}{2}f_n(3(\frac{1}{3}-)) = \frac{1}{2}f_n(1-) = \frac{1}{2}f_n(1) = \frac{1}{2}$ since f_n is continuous at 1, so f_{n+1} is left-continuous at $\frac{1}{3}$, so f_{n+1} is continuous at $\frac{1}{3}$. The proof for $\frac{2}{3}$ is similar and is left as an exercise.

Now,

$$\begin{aligned}
\|f_{n+1}-f_n\|_\infty &= \max\left(\left\|(f_{n+1}-f_n)\Big|_{x\in[0,\frac{1}{3})}\right\|_\infty, \left\|(f_{n+1}-f_n)\Big|_{x\in[\frac{1}{3},\frac{2}{3}]}\right\|_\infty, \left\|(f_{n+1}-f_n)\Big|_{x\in(\frac{2}{3},1]}\right\|_\infty\right) \\
&= \max\left(\left\|(f_{n+1}-f_n)\Big|_{x\in[0,\frac{1}{3})}\right\|_\infty, 0, \left\|(f_{n+1}-f_n)\Big|_{x\in(\frac{2}{3},1]}\right\|_\infty\right) \\
&= \max\left(\left\|\left(\frac{1}{2}f_n(3\cdot)-\frac{1}{2}f_{n-1}(3\cdot)\right)\Big|_{x\in[0,\frac{1}{3})}\right\|_\infty, \left\|\left(\frac{1}{2}f_n(3\cdot-2)-\frac{1}{2}f_{n-1}(3\cdot-2)\right)\Big|_{x\in(\frac{2}{3},1]}\right\|_\infty\right) \\
&\leq \frac{1}{2}\|f_n-f_{n-1}\|_\infty
\end{aligned}$$

So, for some $c, d \in \mathbb{R}$,

$$\|f_{n-1}-f_n\|_\infty \leq \frac{1}{2}\|f_n-f_{n-1}\|_\infty \leq \frac{1}{2^2}\|f_{n-1}-f_{n-2}\|_\infty \leq \dots \leq \frac{c}{2^{n+1}}\|f_1-f_0\|_\infty \leq \frac{d}{2^{n+1}}$$

The constants are introduced to correct for f_0.

Now, fix $n > m$, then, for some $D \in \mathbb{R}$,

$$\begin{aligned}
0 \leq \|f_n-f_m\|_\infty &= \|f_n-f_{n+1}+f_{n+1}-f_{n-2}+\dots+f_{m-1}-f_m\|_\infty \\
&\leq \|f_n-f_{n-1}\|_\infty + \|f_{n-1}-f_{n-2}\|_\infty + \dots + \|f_{m+1}-f_m\|_\infty \\
&\leq D\left(\frac{1}{2^n}+\frac{1}{2^{n-1}}+\dots+\frac{1}{2^{m+1}}\right) \leq \frac{D}{2^{m+1}}\sum_{k=0}^{n-m-1}\frac{1}{2^k} \\
&\leq \frac{D}{2^{m+1}}\sum_{k=0}^{\infty}\frac{1}{2^k} = \frac{D}{2^m} \to 0 \text{ as } m \to \infty
\end{aligned}$$

Since $\|f_n-f_m\|_\infty$ can be made arbitrarily small by taking $n > m$ large enough, (f_n) is uniformly Cauchy. Therefore, $f = \lim_{n\to\infty} f_n$ exists and is continuous, since a uniformly Cauchy sequence converges uniformly and f_n is continuous $\forall n \in \mathbb{N}$. The limit is called Cantor's function or the Devil's Staircase.

Now, $f \in C[0,1]$, so $f \in R[0,1]$. Taking the limit of $f_n(x)$, it is clear that

$$f(x) = \begin{cases} \frac{1}{2}f(3x) & x \in [0, \frac{1}{3}) \\ \frac{1}{2} & x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{2} + \frac{1}{2}f(3x-2) & x \in (\frac{2}{3}, 1] \end{cases}$$

So

$$\int_0^1 f = \int_0^{\frac{1}{3}} f + \int_{\frac{1}{3}}^{\frac{2}{3}} f + \int_{\frac{2}{3}}^1 f = \int_0^{\frac{1}{3}} \frac{1}{2}f(3x)\ dx + \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{1}{2}\ dx + \int_{\frac{2}{3}}^1 \frac{1}{2} + \frac{1}{2}f(3x-2)\ dx$$

Let $y = 3x,\ z = 3x - 2$. Then

$$\int_0^1 f = \frac{1}{6} + \frac{1}{6}\int_0^1 f(y)\ dy + \frac{1}{6} + \frac{1}{6}\int_0^1 f(z)\ dz = \frac{1}{3} + \frac{1}{3}\int_0^1 f, \text{ so } \frac{2}{3}\int_0^1 f = \frac{1}{3}, \text{ so } \int_0^1 f = \frac{1}{2} \quad \blacksquare$$

Remark: Assume that f is constant on $E \subset [0,1]$. Then f is constant on $(\frac{1}{2}, \frac{2}{3}) \cup \frac{1}{3}E \cup (\frac{2}{3} + \frac{1}{3}E)$, so $|E| = \frac{1}{3} + \frac{1}{3}|E| + \frac{1}{3}|E|$, so $\frac{1}{3}|E| = \frac{1}{3}$, so $|E| = 1$, so f is locally constant on some $E \subset [0,1]$ of length 1, but $f : [0,1] \to [0,1]$ continuously.

Note: It is possible to map $[0,1] \subset \mathbb{R}$ onto (surjectively) $[0,1] \times [0,1] \subset R^2$ continuously. If $\gamma : [0,1] \to [0,1] \times [0,1]$ is such a map, then, $\forall x \in [0,1] \times [0,1]$, $\exists t \in [0,1]$ such that $\gamma(t) = x$. A surjective map is relatively simple to construct, such as $\tau : [0,1] \to [0,1] \times [0,1]$, with $\tau : 0.x_1x_2x_3\ldots \mapsto (0.x_1x_3x_5\ldots,\ 0.x_2x_4x_6\ldots)$, but τ is not continuous. A continuous map satisfying these criteria is called a *space filling curve*, and examples were constructed by Hilbert and Peano in the 1890s.

7.2 Uniform Convergence & Integration

Note: Throughout this section, let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$, and $(x, t) \in D$.

Definition 10: A function $f : D \to \mathbb{R}$ is *continuous* at $(x_0, t_0) \in D$ if, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon(x_0, t_0) > 0$ such that, $\forall (x_0, t_0) \in D$ with $|x - x_0| < \delta_\varepsilon(x_0, t_0)$ and $|t - t_0| < \delta_\varepsilon(x_0, t_0)$, $\left|f(x, t) - f(x_0, t_0)\right| < \varepsilon$. An equivalent definition, which shall be introduced later, uses Euclidean distance.

Definition 10.1: A function $f : D \to \mathbb{R}$ is *uniformly continuous* on D if, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon$ such that, $\forall (x, t), (x_0, t_0) \in D$ with $|x - x_0| < \delta_\varepsilon$ and $|t - t_0| < \delta_\varepsilon$, $\left|f(x, t) - f(x_0, t_0)\right| < \varepsilon$.

Lemma 26.2: If f is continuous on D, then it is also uniformly continuous if D is closed.

Proof: The proof is left as an exercise. *Hint:* Fix $t \in [c, d]$, and consider $f(\cdot, t) : [a, b] \to \mathbb{R}$ with $f : x \mapsto f(x, t)$. ■

Exercise: Prove that, if f is continuous on D, then $f(\cdot, t)$ is continuous on $[a, b]$ (so $f(\cdot, t) \in C[a, b]$).

Lemma 26.1: If a map I is such that

$$I : t \in [c, d] \mapsto \int_a^b f(x, t)\, dx$$

then, if $f \in C(D)$, $I \in C[c, d]$ (I is often called *the integral depending on a parameter*).

Proof: Let $(t_n)_{n \geq 1} \subset [c, d]$ with $\lim_{n\to\infty} t_n = t_0$, and let $f_n : [a, b] \to \mathbb{R}$ with $f_n : x \mapsto f(x, t_n)$. Now, f is continuous on D, so f is uniformly continuous on D since D is closed, so, $\forall \varepsilon > 0$, $\exists \delta_{\varepsilon > 0}$ such that, if $|x - x_0| < \delta_\varepsilon$ and $|t - t_0| < \delta_\varepsilon$, then $\left|f(x,\ t) - f(x_0, t_0)\right| < \varepsilon$. Since $(t_n), (t_{n+m}) \to t_0$ as $n \to \infty$, then, eventually in n, $|t_n - t_{n+m}| < \delta_\varepsilon$, so $\left|f_n(x) - f_{n+m}(x)\right| = \left|f(x, t_n) - f(x, t_{n+m})\right| < \varepsilon$ eventually in n, so (f_n) is uniformly Cauchy, so (f_n) converges uniformly to f. Now,

$$\lim_{n\to\infty} I(t_n) = \lim_{n\to\infty} \int_a^b f(x, t_n)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx = \int_a^b f(x, t_0) = I(t_0)$$

So I is continuous at t_0, and since $t_0 \in [c, d]$ was arbitrary, I is continuous on $[c, d]$. ■

Proposition 27: If f, $\frac{\partial f}{\partial t}$ are continuous on $[a, b] \times [c, d]$, then, $\forall t \in (c, d)$,

$$\frac{\partial}{\partial t} \int_a^b f(x, t)\, dx = \int_a^b \frac{\partial f}{\partial t}(x, t)\, dx$$

Or, alternatively, $\forall t \in (c, d)$,

$$F(t) = \int_a^b f(x, t)\, dx \text{ and } G(t) = \int_a^b \frac{\partial f}{\partial t}(x, t)\, dx$$

both exist on (c, d), F is differentiable on (c, d), and $F' = G$.

Proof: Fix $t \in (c,d)$. Then $f(\cdot,t), \frac{\partial f}{\partial t}(\cdot,t) \in C[a,b]$, so F and G exist. Now, $\exists [c_1,d_1] \subset [c,d]$ such that $t \in (c_1,d_1)$. But $\frac{\partial f}{\partial t}$ is continuous on $[a,b]\times[c,d]$, and so is uniformly continuous on $[a,b]\times[c,d]$. Then

$$\left|\frac{F(t+h)-F(t)}{h} - G(t)\right| = \left|\int_a^b \frac{f(x,t+h)-f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t)\, dx\right|$$
$$\stackrel{\text{MVT}}{=} \left|\int_a^b \frac{\partial f}{\partial \tau}(x,\tau) - \frac{\partial f}{\partial t}(x,t)\, dx\right|$$

for some $\tau \in (t,t+h)$. Now, $\frac{\partial f}{\partial \tau}$ is uniformly continuous on $[a,b]\times[c,d]$, so $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $\forall h$ with $|h| < \delta_\varepsilon$,

$$\left|\frac{\partial f}{\partial \tau}(x,\tau) - \frac{\partial f}{\partial t}(x,t)\right| < \varepsilon, \text{ so} \left|\int_a^b \frac{\partial f}{\partial \tau}(x,\tau) - \frac{\partial f}{\partial t}(x,t)\, dx\right| < \int_a^b \varepsilon = \varepsilon(b-a)$$

so

$$\lim_{h\to 0}\left|\frac{F(t+h)-F(t)}{h}\right| = G(t)$$

so $F'(t) = G(t)$. ■

Theorem 28 - Fubini's Theorem: If $f : D \to \mathbb{R}$ is continuous, then

$$\int_a^b \left(\int_c^d f(x,y)\, dy\right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx\right) dy$$

Proof: Let

$$F(t) = \int_a^t \left(\int_c^d f(x,y)\, dy\right) dx - \int_c^d \left(\int_a^t f(x,y)\, dx\right) dy$$

for $t \in [a,b]$. Now, $F(a) = 0$, and

$$F'(t) \stackrel{\text{FTC}}{=} \int_c^d f(t,y)\, dy - \int_c^d f(t,y)\, dy = 0$$

F is continuous on $[a,b]$ and differentiable on (a,b), and $F'(t) = 0$, so $F(b) - F(a) = 0$, so

$$\int_a^b \left(\int_c^d f(x,y)\, dy\right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx\right) dy \quad ■$$

Note: f must be continuous on the whole of D, or counterexamples such as the following can arise: Let $D = [0,2]\times[0,1]$, and let $f(x,y) : D \to \mathbb{R}$ with

$$f : (x,y) \mapsto \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Then

$$\int_0^2 \left(\int_0^1 f(x,y)\, dy\right) dx = \frac{1}{5} \neq \frac{1}{20} = \int_0^1 \left(\int_0^2 f(x,y)\, dx\right) dy$$

7.3 Uniform Convergence & Differentiation

Note: Suppose that $(f_n)_{n\geq 1}$ is a sequence of functions on $[a,b]$, and that $(f_n) \to f$ uniformly as $n \to \infty$. Then, even if every f_n is differentiable on $[a,b]$, f is not necessarily differentiable on $[a,b]$. And even if f is smooth, (f_n') does not necessarily converge to f'.

Example: Suppose that $f_n(x) = \frac{1}{n}\cos nx$ on $\mathbb{R}$. This function is smooth, and $f_n'(x) = \sin nx$. Let $f : x \mapsto 0$. Then $\|f_n - f\|_\infty = \|f_n\|_\infty = \frac{1}{n}\|\cos n\cdot\|_\infty = \frac{1}{n} \to 0$ as $n \to \infty$, so $(f_n) \to f$ uniformly as $n \to \infty$. All f_ns are smooth, and f is smooth, since $f' \equiv 0$. But $(f_n') \not\to 0$ as $n \to \infty$, in fact, (f_n') does not converge at all, except for at certain points such as $x = k\pi$ for $k \in \mathbb{N}$.

Example: Let $f_n : \mathbb{R} \to \mathbb{R}$, with $f_n : x \mapsto \sqrt{x^2 + \frac{1}{n}}$. Then $(f_n(x)) \to |x|$ pointwise as $n \to \infty$. Let $f(x) = |x|$. Then $\left|f_n(x)^2 - f(x)^2\right| = \left|x^2 + \frac{1}{n} - x^2\right| = \frac{1}{n}$. $\left|f_n(x) - f(x)\right| = \frac{\left|f_n(x)^2 - f(x)^2\right|}{\left|f_n(x) + f(x)\right|} \leq \frac{1}{n}\frac{1}{\sqrt{\frac{1}{n}}} = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so $(f_n) \to f$ uniformly as $n \to \infty$. Now, all f_ns are smooth, and $(f_n) \to f$ uniformly as $n \to \infty$, but f is not differentiable at 0.

Note: If a function $f : [a,b] \to \mathbb{R}$ is continuously differentiable, it is called a C^1 function, or, alternatively, $f \in C^1[a,b]$ is written, where $C^1[a,b]$ is the space of continuously differentiable functions on $[a,b]$.

Theorem 29: Let (f_n) be a sequence of C^1 functions on $[a,b]$, with $(f_n) \to f$ pointwise as $n \to \infty$ on $[a,b]$, and suppose that (f_n') converges uniformly. Then $f \in C^1[a,b]$ and $\lim_{n\to\infty}(f_n') = f'$.

Proof: Let $g = \lim_{n\to\infty}(f_n')$. Now, $\forall n \in \mathbb{N}$, $f_n' \in C[a,b]$, so $g \in C[a,b]$ since $(f_n') \to g$ uniformly, so

$$\int_a^x g = \int_a^x \lim_{n\to\infty}(f_n') = \lim_{n\to\infty}\int_a^x f_n' \stackrel{\text{FTC}}{=} \lim_{n\to\infty}[f_n(x) - f_n(a)] = f(x) - f(a)$$

so, since

$$f(x) = f(a) + \int_a^x g$$

$\int_a^x g$ is continuous, so by the FTC, $f' = g = \lim_{n\to\infty}(f_n)$, so

$$\left(\lim_{n\to\infty}(f_n)\right)' = f' \stackrel{\text{FTC}}{=} \lim_{n\to\infty}(f_n') \quad \blacksquare$$

8 Functional Series

8.1 The Weierstrass M-Test & Other Useful Results

Note: Let $(f_n)_{n\geq 1}$ be a sequence of functions on $A \subset \mathbb{R}$. Then thei functional series $\sum_{k=1}^{\infty} f_k$ converges pointwise (or uniformly) on A if the sequence of partial sums $S_n = (\sum_{k=1}^{n} f_k)_{n\geq 1}$ converges pointwise (or uniformly).

Theorem 24′: Let $(f_n)_{n\geq 1} \subset R[a,b]$. Suppose that $S_n = \sum_{k=1}^{n} f_k$ converges uniformly. Then $\sum_{k=1}^{\infty} f_k \in R[a,b]$ and

$$\int_a^b \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_a^b f_k$$

Proof: If $(f_k) \in R[a,b]$, then $S_n = \sum_{k=1}^{n} f_k \in R[a,b]\ \forall n \in \mathbb{N}$, so $\int_a^b \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_a^b f_k$. ■

Theorem 25′: Suppose that $(f_n)_{n\geq 1} \subset C[a,b]$ such that $S_n = \sum_{k=1}^{n} f_k$ converges uniformly. Then $\sum_{k=1}^{\infty} f_k \in C[a,b]$, and, $\forall x \in [a,b]$,

$$\lim_{x\to x_0} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \lim_{x\to x_0} f_k(x)$$

Proof: $S_n \in C[a,b]\ \forall n \in \mathbb{N}$, so $\lim_{x\to x_0} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \lim_{x\to x_0} f_k(x)$. ■

Theorem 29′: If $(f_n)_{n\geq 1} \subset C^1[a,b]$ and $S_n' = \sum_{k=1}^{n} f_k'$ converges uniformly on $[a,b]$, then

$$\frac{d}{dx} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x)$$

Proof: The proof is left as an exercise. ■

Theorem 30 - Weierstrass M-Test: Let $(f_n)_{n\geq 1}$ be a sequence of functions on $A \subset \mathbb{R}$. Then, if $\exists (M_k)_{k\geq 1} \subset \mathbb{R}$ such that, $\forall x \in A$, $|f_n(x)| < M_n$ and $\sum_{k=1}^{\infty} M_k$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Proof: $\sum_{k=1}^{\infty} M_k$ converges, so $(\sum_{k=1}^{n} M_k)_{n\geq 1}$ is Cauchy, so, $\forall \varepsilon_n > 0\ \exists N_{\varepsilon_n}$ such that $\forall n, m > N_{\varepsilon_n}$, with $n > m$, $\left|\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k\right| < \varepsilon_n$. Then $\left|\sum_{k=m+1}^{n} M_k\right| < \varepsilon_n$, so, since $M_K \geq 0$, $\sum_{k=m+1}^{n} M_k < \varepsilon_n$. Therefore

$$\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon_n$$

This holds $\forall x \in A$, so the sequence of partial sums of f_k is uniformly Cauchy, so $\sum_{k=1}^{\infty} f_k$ converges uniformly to its pointwise limit. ■

Corollary 30.1: If $(f_n)_{n\geq 1}$ is such that $\sum_{k=1}^{\infty} \|f_k\|_{\infty}$ converges, then $\sum_{k=1}^{\infty} f_k$ converges uniformly.

8.2 Taylor & Fourier Series

Motivation: Taylor polynomials are of the form

$$S_n = \sum_{k=0}^{n} \left[\frac{1}{k!} f^{(k)}(a)(x-a)^k\right]$$

and were studied in Analysis II.

Fourier series are of the form

$$\frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx),\ x \in [0, 2\pi]$$

Applications of functional series include solving PDEs, ODEs and time series.

Definition: For any $f \in R[-\pi, \pi]$, the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

with coefficients given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\ dx,\ k \geq 0$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\ dx,\ k > 0$$

is called the *Fourier series generated by* f.

Note: It is not immediately obvious for which classes of functions this series is convergent, or if it converges to f.

Theorem 31: Let $(a_k)_{k\geq 0}, (b_k)_{k\geq 1} \subset \mathbb{R}$, with $\sum_{k=0}^{\infty}|a_k|$ and $\sum_{k=1}^{\infty}|b_k|$ convergent. Then

- $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ converges uniformly on $\mathbb{R}$.
- The pointwise limit $f : \mathbb{R} \to \mathbb{R}$ is continuous.
- f is 2π-periodic ($f(x + 2\pi) = f(x)$).

Moreover,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\ dx,\ k \geq 0$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\ dx,\ k > 0$$

so the terms of the series can be recovered.

Proof (Convergence): $\forall x \in \mathbb{R}$, $|a_k \cos kx + b_k \sin kx| \leq |a_k \cos kx| + |b_k \sin kx| \leq |a_k| + |b_k| := M_k$. Then $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty}|a_k| + |b_k| = \sum_{k=1}^{\infty}|a_k| + \sum_{k=1}^{\infty}|b_k|$, so $\sum_{k=1}^{\infty} M_k$ converges, so the Fourier series converges by the M-test.

Proof (Continuity): Let $f(x) := \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$. Then $f \in C(\mathbb{R})$.

Proof (Periodicity): Let $S_N(y) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$. Then $f(x + 2\pi) - f(x) = \lim_{n\to\infty} S_n(x+2\pi) - \lim_{n\to\infty} S_n(x)$. Both of these limits exist, so $f(x+2\pi) - f(x) = \lim_{n\to\infty} S_n(x + 2\pi) - S_n(x) = 0$, since $\cos(k(x + 2\pi)) = \cos kx$ and $\sin(k(x + 2\pi)) = \sin kx$.

Lemma: $\forall m \in \mathbb{Z}$

$$\int_{-\pi}^{\pi} e^{imx} dx = 2\pi \mathbb{1}(m=0)$$

where $\mathbb{1}(m=0)$ is the indicator function for $m=0$.

Proof: For $m \in \mathbb{Z} \setminus \{0\}$:

$$\int_{-\pi}^{\pi} e^{imx}\, dx = \int_{-\pi}^{\pi} \cos mx + i \sin mx\, dx = \int_{-\pi}^{\pi} \cos mx\, dx + i \int_{-\pi}^{\pi} \sin mx\, dx = \frac{2 \sin \pi m}{m} = 0$$

For $m=0$:

$$\int_{-\pi}^{\pi} e^{imx}\, dx = \int_{-\pi}^{\pi} e^{0}\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi$$

So

$$\int_{-\pi}^{\pi} e^{imx}\, dx = 2\pi \mathbb{1}(m=0) \quad \blacksquare$$

Proof (Recovery):

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\, dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)\, dx \\ &= \frac{a_0}{2} \int_{-\pi}^{\pi} 1\, dx + \sum_{k=1}^{\infty} \left[a_k \int_{-\pi}^{\pi} \cos kx\, dx + b_k \int_{-\pi}^{\pi} \sin kx\, dx \right] = \pi a_0\end{aligned}$$

so

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 0x\, dx$$

Now,

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx\, dx + \sum_{k=1}^{\infty} \left[a_k \int_{-\pi}^{\pi} \cos kx \cos mx\, dx \right] + \sum_{k=1}^{\infty} \left[b_k \int_{-\pi}^{\pi} \sin kx \cos mx\, dx \right]$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} \sin kx \cos mx\, dx &= \int_{-\pi}^{\pi} \frac{1}{2i} (e^{ikx} - e^{-ikx}) \frac{1}{2} (e^{imx} - e^{-imx})\, dx \\ &= \frac{1}{4i} \int_{-\pi}^{\pi} e^{i(k-m)x} - e^{i(m-k)x} + e^{i(k+m)x} - e^{i(m+k)x}\, dx \\ &= \frac{1}{4i} (2\pi \mathbb{1}(k-m=0) - 2\pi \mathbb{1}(m-k=0)) = 0\end{aligned}$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} \cos kx \cos mx\, dx &= \int_{-\pi}^{\pi} \frac{1}{2} (e^{ikx} + e^{-ikx}) \frac{1}{2} (e^{imx} - e^{-imx})\, dx \\ &= \frac{1}{4} \int_{-\pi}^{\pi} e^{i(k-m)x} + e^{i(m-k)x} + e^{i(k+m)x} + e^{i(m+k)x}\, dx \\ &= \frac{1}{4} 2\pi ((\mathbb{1}(k-m=0) + \mathbb{1}(m-k=0)) = \pi \mathbb{1}(k=m)\end{aligned}$$

then

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \sum_{k=1}^{\infty} (a_k \pi \mathbb{1}(k=m)) = \pi a_m$$

so

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx$$

The proof for b_k is similar and so is left as an exercise. $\blacksquare$

Note: This proof relies on an application of Theorem 24, where the sum is multiplied by a bounded function. The proof that the result holds is left as an exercise.

Corollary 31.1: If $f : [-\pi, \pi] \to \mathbb{R}$ is a C^2 function, (or $f' \in C[-\pi, \pi]$), and $f(\pi) = f(-\pi)$. Then the Fourier series generated by f converges to f uniformly.

Note: It must be established that

(i) The Fourier series converges uniformly.

(ii) The limit is f.

However, (ii) is a fairly clunky argument using existing tools, and so shall be omitted and left as an exercise, whilst (i) shall be proved.

Proof:

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\;dx = \frac{1}{\pi k}\int_{-\pi}^{\pi} f(x)\cos' kx\;dx = \frac{1}{\pi k}\left(\left[f(x)\cos kx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\cos kx\;dx\right) \\ &= \frac{1}{\pi k}\int_{-\pi}^{\pi} f'(x)\cos kx\;dx = \frac{1}{\pi k^2}\int_{-\pi}^{\pi} f'(x)\sin' kx\;dx = \frac{1}{\pi k^2}\left(\left[f'(x)\sin kx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f''(x)\sin kx\;dx\right) \\ &= -\frac{1}{\pi k^2}\int_{-\pi}^{\pi} f''(x)\sin kx\;dx \end{aligned}$$

So

$$|b_k| = \frac{1}{\pi k^2}\left|\int_{-\pi}^{\pi} f(x)\sin kx\;dx\right| \le \frac{1}{\pi k^2}\left\|f''\cdot\sin k\cdot\right\|_\infty \cdot 2\pi = \frac{2}{k^2}\left\|f''\right\|_\infty := \frac{c_b}{k^2}$$

and c_b is independent of k. A similar argument establishes that $|a_k| \le \frac{c_a}{k^2}$ for $k > 0$, where c_a is also independent of k. $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges, so the Fourier series converges by the M-test with $M_k = \frac{c_a+c_b}{k^2}$ for $k > 0$. ■

Example (Temperature Equilibration on a Non-Uniformly Heated Ring): Consider the unit circle S^1. Let any point on it be described by the angle $\varphi \in [-\pi, \pi]$ it makes with the x-axis, and let the initial temperature on the ring be given by $f(\varphi) \in C^2[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$. Then the temperature T at time t at the point at φ, for $t > 0$, $\varphi \in [-\pi, \pi]$, is given by the *Fourier heat equation*

$$\frac{\partial T}{\partial t}(t,\varphi) - \frac{\partial^2 T}{\partial \varphi^2}(t,\varphi) = 0$$

Since the point at π is the same as the point at $-\pi$, there are some boundary conditions to consider also:

$$\begin{aligned} T(t,-\pi) &= T(t,\pi) \\ \frac{\partial T}{\partial \varphi}(t,-\pi) &= \frac{\partial T}{\partial \varphi}(t,\pi) \end{aligned}$$

The initial condition is given by

$$T(0,\varphi) = f(\varphi)$$

Throughout this example,

$$\dot{T} := \frac{\partial T}{\partial t},\; T' := \frac{\partial T}{\partial \varphi},\; T'' := \frac{\partial^2 T}{\partial \varphi^2}$$

Now, $(\cos k\varphi)' = -k\sin k\varphi$, and $(\cos k\varphi)'' = -k^2\cos k\varphi$, so $A(t)\cos k\varphi$ seems a suitable ansatz to make for a solution. Substituting this into the heat equation yields $\dot{A}\cos k\varphi + k^2 A\cos k\varphi = 0$, so $\dot{A}(t) + k^2 A(t) = 0$, which is an ODE. The boundary conditions are in fact satisfied by either this solution, or one using the ansatz $B(t)\sin k\varphi$, but neither of these satisfy the initial conditions. Since a finite linear combination of any of these functions will have the same shortcoming, it seems sensible to take the next ansatz as being an infinite linear combination of them, such as

$$T(t,\varphi) = \frac{a_0(t)}{2} + \sum_{k=1}^{\infty}(a_k(t)\cos k\varphi + b_k(t)\sin k\varphi)$$

Assume that T, $\dot{T}$, T' and T'' all converge uniformly for $t > 0$, $\varphi \in [-\pi,\pi]$. Then substituting T into the heat equation and differentiating pointwise yields

$$\frac{\dot{a_0}}{2} + \sum_{k=1}^{\infty}(\dot{a_k}\cos k\varphi + \cdot b_k \sin k\varphi) + \sum_{k=1}^{\infty}(a_k k^2\cos k\varphi + b_k k^2 \sin k\varphi) = 0$$

So

$$\frac{\dot{a_0}}{2} + \sum_{k=1}^{\infty}\left[(\dot{a_k} + a_k k^2)\cos k\varphi + (\dot{b_k} + b_k k^2)\sin k\varphi\right] = 0$$

which is the Fourier series generated by 0, so

$$\begin{aligned} \dot{a_0} &= 0 \\ \dot{a_k} + k^2 a_k &= 0 \\ \dot{b_k} + k^2 b_k &= 0 \end{aligned}$$

for $k > 0$. This yields infinitely many ODEs, which require infinitely many initial conditions.

Expanding $f(\varphi) \in C^2[-\pi,\pi]$ by a Fourier series yields

$$f(\varphi) = \frac{a_0^{(f)}}{2} + \sum_{k=1}^{\infty}\left(a_k^{(f)}\cos k\varphi + b_k^{(f)}\sin k\varphi\right)$$

where $n_k^{(f)}$ denotes the coefficient n_k dependent on the function f. This gives the required amount of initial conditions. Then

$$\begin{aligned} a_0(t) &= a_0^{(f)} \\ a_k(t) &= a_k^{(f)} e^{-k^2 t} \\ b_k(t) &= b_k^{(f)} e^{-k^2 t} \end{aligned}$$

So

$$T(t,\varphi) = \frac{a_0^{(f)}}{2} + \sum_{k=1}^{\infty} e^{-k^2 t}\left(a_k^{(f)}\cos k\varphi + b_k^{(f)}\sin k\varphi\right)$$

Whilst this is a solution, its uniqueness must also be established, however that shall not be done here.

Exercise: Check that T converges uniformly for $t \geq 0$, and that $\dot{T}$, T', T'' do so also for $t > 0$.

Note: Over time, the temperature becomes distributed evenly across the whole ring, since

$$\lim_{t\to\infty} T(t,\varphi) = \frac{a_0^{(f)}}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\, d\varphi = \text{Average Temperature.}$$

The approach to this equilibrium is exponential.

Part II

Norms

9 Normed Vector Spaces

9.1 Definition & Basic Properties of a Norm

Definition 11: Let V be a vector space over $\mathbb{R}$. Then a function $\|\cdot\| : V \to \mathbb{R}$, with $\|\cdot\| : \underline{v} \in V \mapsto \|\underline{v}\|$ which satisfies the following:

(i) $\forall \underline{v} \in V, \|\underline{v}\| \geq 0$ (*positivity*). Moreover, $\|\underline{v}\| = 0 \iff \underline{v} = \underline{0} \in V$ (the ability to *separate points*).

(ii) $\forall \lambda \in \mathbb{R}, \forall \underline{v} \in V, \|\lambda \underline{v}\| = |\lambda| \cdot \|\underline{v}\|$ (*absolute homogeneity*).

(iii) $\forall \underline{u}, \underline{v} \in V, \|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ (*triangle inequality*).

is called a *norm on V*. A pair $(V, \|\cdot\|)$ is called a *normed vector space*, or a *normed space.*

Remarks:

- (ii) and (iii) use the linear structure of V.
- Positivity follows from (ii) and (iii), since, $\forall \underline{v} \in V$, $0 = 0 \cdot \|\underline{v}\| \overset{\text{abs.}}{\underset{\text{hom.}}{=}} \|0 \cdot \underline{v}\| - \|\underline{v} - \underline{v}\| \leq \|\underline{v}\| + \|-\underline{v}\| = \|\underline{v}\| + |-1| \cdot \|\underline{v}\| = \|\underline{v}\| + \|\underline{v}\| = 2\|\underline{v}\|$, so $0 \leq \|\underline{v}\|$.

Examples:

1. $|\cdot|$, the absolute value, is a norm on $\mathbb{R}$. The triangle inequality is satisfied since $|x + y| \leq |x| + |y|$. The proofs of the other properties are left as an exercise.
2. $\|\cdot\|_\infty$, the sup norm, is a norm on $B[a, b]$.
   (i) : $\forall f \in B[a, b]$, if $\|f\|_\infty = 0$, then $\sup_{x\in[a,b]}\big|f(x)\big| = 0$, so, $\forall x \in [a, b]$, $0 \leq f(x) \leq 0$, so $f \equiv 0$ on $[a, b]$, so $\|\cdot\|_\infty$ can separate points ($\|f - g\|_\infty = 0 \iff f = g, \|f - g\|_\infty > 0 \iff f \neq g$).
   (ii) : $\forall \lambda \in \mathbb{R}, \forall f \in B[a, b], \|\lambda f\|_\infty = \sup_{x\in[a,b]}\big|\lambda f(x)\big| = \sup_{x\in[a,b]}|\lambda|\big|f(x)\big| = |\lambda| \sup_{x\in[a,b]}\big|f(x)\big| = \lambda\|f\|_\infty$, so $\|\cdot\|_\infty$ has absolute homogeneity.
   (iii) $\forall f, g \in B[a, b], \|f + g\|_\infty = \sup_{x\in[a,b]}\big|f(x) + g(x)\big| = \sup_{x\in[a,b]}\big(\big|f(x)\big| + \big|g(x)\big|\big) \leq \sup_{x\in[a,b]}\big|f(x)\big| + \sup_{x\in[a,b]}\big|g(x)\big| = \|f\|_\infty + \|g\|_\infty$, so $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$, so $\|\cdot\|_\infty$ satisfies the triangle equality.

Note: Norms generalise the notion of magnitude of vectors in $\mathbb{R}^n$.

Proposition 34: Let $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (a *vector*). Then the following functions on $\mathbb{R}^n$ are norms:

1. $$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i| \quad \text{(the } \textit{taxicab} \text{ or } \textit{Manhattan} \text{ norm)}$$
2. $$\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} {x_i}^2} \quad \text{(the } \textit{Euclidean} \text{ norm or } \textit{Euclidean distance}\text{)}$$
3. $$\|\underline{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$$

Proof: Triangle inequality:

1.
$$\|\underline{u}+\underline{v}\|_1 = \sum_{i=1}^{n}|u_i+v_i| \leq \sum_{i=1}^{n}(|u_i|+|v_i|) = \sum_{i=1}^{n}|u_i| + \sum_{i=1}^{n}|v_i| = \|\underline{u}\|_1 + \|\underline{v}\|_1$$

2. For any $\underline{u}, \underline{v} \in \mathbb{R}^n$,
$$\underline{u}\cdot\underline{v} = \sum_{i=1}^{n} u_i \cdot v_i = \|\underline{u}\|_n \cdot \|\underline{v}\|_2 \cdot \cos\theta \leq \|\underline{u}\|_2 \cdot \|\underline{v}\|_2$$
for some $\theta \in [0, 2\pi]$. This is known as the *Cauchy-Schwarz inequality for* $\mathbb{R}^n$. Then
$$\begin{aligned}\|\underline{u}+\underline{v}\|_2^2 &= (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) \\ &= \sum_{i=1}^{n}(u_i+v_i)\cdot(u_i+v_i) \\ &= \underline{u}\cdot(\underline{u}+\underline{v}) + \underline{v}\cdot(\underline{u}+\underline{v}) \\ &\underset{\text{Schwarz}}{\overset{\text{Cauchy}}{\leq}} \|\underline{u}\|_2\cdot\|\underline{u}+\underline{v}\|_2 + \|\underline{v}\|_2\cdot\|\underline{u}+\underline{v}\|_2\end{aligned}$$
So
$$\|\underline{u}+\underline{v}\|_2^2 \leq \|\underline{u}\|_2\cdot\|\underline{u}+\underline{v}\|_2 + \|\underline{v}\|_2\cdot\|\underline{u}+\underline{v}\|_2$$
The case where $\underline{u}+\underline{v}=0$ is trivial to check, otherwise
$$\|\underline{u}+\underline{v}\|_2 \leq \|\underline{u}\|_2 + \|\underline{v}\|_2$$

3.
$$\|\underline{u}+\underline{v}\|_\infty = \max_{1\leq i\leq n}|u_i+v_i| \leq \max_{1\leq i\leq n}(|u_i|+|v_i|) \leq \max_{1\leq i\leq n}|u_i| + \max_{1\leq i\leq n}|v_i| = \|\underline{u}\|_\infty + \|\underline{v}\|_\infty$$

The verification of the other properties is left as an exercise. ■

Remark: All of these norms are instances of the following family of norms:
$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}, \; p = 1, 2, \ldots$$
In particular,
$$\|\underline{x}\|_\infty = \lim_{p\to\infty}\|\underline{x}\|_p$$
That this family of functions is a family of norms follows from the *Minkowski inequality*:
$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}} \quad \forall \underline{x}, \underline{y} \in \mathbb{R}^n$$

Note : $\mathbb{R}^n$ can be thought of as the space of functions on $N_n = \{1, 2, \ldots, n\}$ by considering $\underline{x} : N_n \to \mathbb{R}$, with $\underline{x} : j \mapsto x_j$.

9.2 Equivalence & Banach Spaces

Definition 12: Norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on V are called *equivalent* (or *Lipschitz equivalent*) if $\exists k_1, k_2 \geq 0$ such that, $\forall \underline{v} \in V$, $k_1 \cdot \|\underline{v}\|_b \leq \|\underline{v}\|_a \leq k_2 \cdot \|\underline{v}\|_b$. This equivalence is denoted by $\|\cdot\|_a \sim \|\cdot\|_b$.

Remark: The equivalence of norms is an equivalence relation on the set of all norms on V. That is,

1. $\|\cdot\|_a \sim \|\cdot\|_a$ (choose $k_1 = k_2 = 1$).
2. $\|\cdot\|_a \sim \|\cdot\|_b \iff \|\cdot\|_b \sim \|\cdot\|_a$.
3. If $\|\cdot\|_a \sim \|\cdot\|_b$ and $\|\cdot\|_b \sim \|\cdot\|_c$, then $\|\cdot\|_a \sim \|\cdot\|_c$.

The proof is left as an exercise.

Lemma 35: $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$ are all equivalent on $\mathbb{R}^n$.

Proof: Because equivalence is an equivalence relation, it is enough to prove that $\|\cdot\|_1 \sim \|\cdot\|_\infty$ and $\|\cdot\|_2 \sim \|\cdot\|_\infty$. Now, $\forall \underline{x} \in \mathbb{R}^n$,

$$\|\underline{x}\|_\infty = \max_{1\leq i\leq n} |x_i| \leq \|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} \max_{1\leq j\leq n} |x_j| = n\|\underline{x}\|_\infty$$

So $\|\cdot\|_1 \sim \|\cdot\|_\infty$ with $k_1 = 1$, $k_2 = n$.

There is a similar argument for the second part of the proof, although a much more general statement will be proved shortly, and so this shall be left as an exercise.

Definition 13: Let $(V, \|\cdot\|)$ be a normed space. Then

(i) A sequence $(\underline{\sigma_n})_{n\geq 1} \in V$ *converges* to $\underline{\sigma} \in V$ if $\lim_{n\to\infty} \|\underline{\sigma_n} - \underline{\sigma}\| = 0$.

(ii) A sequence $(\underline{\sigma_n})_{n\geq 1} \in V$ is *Cauchy* if, $\forall \varepsilon > 0$, $\exists N_\varepsilon$ such that, $\forall n, m > N_\varepsilon$, $\|\underline{\sigma_n} - \underline{\sigma_m}\| < \varepsilon$.

(iii) If every Cauchy sequence in V converges, then $(V, \|\cdot\|)$ is called *Banach* (or *complete*). A complete normed space is also called a Banach space.

Remarks:

- Any Cauchy sequence on $(\mathbb{R}, |\cdot|)$ converges, so $(\mathbb{R}, |\cdot|)$ is Banach.
- Consider $(S[a,b], \|\cdot\|_\infty)$. Let $(s_n)_{n\geq 1} \subset S[a,b]$ be a uniform Cauchy sequence, so (s_n) is Cauchy with respect to $\|\cdot\|_\infty$. Then $s_n \to r \in R[a,b]$ as $n \to \infty$, so, in general, (s_n) doesn't converge to a "point" in $S[a,b]$ because not all regulated functions are step functions, so $(S[a,b], \|\cdot\|_\infty)$ is not Banach.

Note: All finite-dimensional normed spaces are complete. A proof for the case when the space is over $\mathbb{R}$ shall be given later.

Remarks:

1. If $\lim_{n\to\infty} \underline{\sigma_n}$ exists, then it is unique.

   Suppose that $\underline{\sigma_n} \to \underline{a} \in V$ and $\underline{\sigma_n} \to \underline{b} \in V$ as $n \to \infty$. Then $0 \leq \|\underline{a} - \underline{b}\| = \|\underline{a} - \underline{\sigma_n} + \underline{\sigma_n} - \underline{b}\| \leq \|\underline{a} - \underline{\sigma_n}\| + \|\underline{\sigma_n} - \underline{b}\| \to 0$ as $n \to \infty$, so $\|\underline{a} - \underline{b}\| = 0$, so $\underline{a} = \underline{b}$ by separation of points.

2. $(R[a,b], \|\cdot\|_\infty)$ is complete, because a uniformly Cauchy sequence of regulated functions converges to a regulated function. Also, $(C[a,b], \|\cdot\|_\infty)$ and $(B[a,b], \|\cdot\|_\infty)$ are complete by results demonstrated earlier.

3. Suppose that $\|\cdot\|_a \sim \|\cdot\|_b$. Then, if $\lim_{n\to\infty} \underline{\sigma_n} = \sigma$ with respect to $\|\cdot\|_a$, then $\lim_{n\to\infty} \underline{\sigma_n} = \sigma$ with respect to $\|\cdot\|_b$. The proof is left as an (important) exercise.
4. If $\|\cdot\|_a \sim \|\cdot\|_b$, then $(V,\|\cdot\|_a)$ is Banach iff $(V,\|\cdot\|_b)$ is Banach. This is essentially a consequence of the previous remark.
5. $(V,\|\cdot\|)$ is Banach iff any series $\sum_{n=1}^{\infty} \underline{\sigma_n}$ which converges absolutely converges to an element in V. $\sum_{n=1}^{\infty} \underline{\sigma_n}$ converges absolutely if $\sum_{n=1}^{\infty} \|\underline{\sigma_n}\|$ converges. The proof of this is left as an exercise, although the $\Longleftarrow$ direction is difficult.

Theorem 36: All norms on $\mathbb{R}^n$ are (Lipschitz) equivalent.

Proof: Let $(\underline{e_1}, \ldots, \underline{e_n}) \subset \mathbb{R}^n$ be the standard basis of $\mathbb{R}^n$. Then, $\forall \underline{x} \in \mathbb{R}^n$, $\underline{x} = \sum_{k=1}^{n} x_k \underline{e_k}$. Now,

$$\|\underline{x}\| = \left\|\sum_{k=1}^{n} x_k \underline{e_k}\right\| \leq \sum_{k=1}^{n} \|x_k \underline{e_k}\| = \sum_{k=1}^{n} |x_k| \|\underline{e_k}\| \leq \max_{1\leq j\leq n} |x_j| \sum_{k=1}^{n} \|\underline{e_k}\| = \|\underline{x}\|_\infty \sum_{k=1}^{n} \|\underline{e_k}\|$$

Let $k_2 = \sum_{k=1}^{n} \|\underline{e_k}\|$. Then $\exists k_2$ such that, $\forall \underline{x} \in \mathbb{R}^n$, $\|\underline{x}\| \leq k_2 \|\underline{x}\|_\infty$.

Now, let

$$J := \inf_{\underline{x}\neq \underline{0}} \frac{\|\underline{x}\|}{\|\underline{x}\|_\infty},\ A := \left\{\frac{\|\underline{x}\|}{\|\underline{x}\|_\infty} : \underline{x} \neq \underline{0}\right\},\ B := \left\{\|\underline{x}\| : \|\underline{x}\|_\infty = 1\right\}$$

Clearly $B \subset A$ by definition. If $\alpha \in A$, then $\exists \underline{u}, \underline{v} \in \mathbb{R}^n$ such that

$$\alpha = \frac{\|\underline{u}\|}{\|\underline{u}\|_\infty} = \left\|\frac{1}{\|\underline{u}\|_\infty} \cdot \underline{u}\right\| = \|\underline{v}\|$$

Then

$$\|\underline{v}\|_\infty = \left\|\frac{\underline{u}}{\|\underline{u}\|_\infty}\right\|_\infty = \frac{1}{\|\underline{u}\|_\infty} \cdot \|\underline{u}\|_\infty = 1$$

so $\alpha \in B$, so $A \subset B$, so $B = A$. Then $J = \inf A = \inf B$, and clearly $J \geq 0$.

Suppose that $J = 0$. Then $\exists \underline{x_1}, \underline{x_2}, \underline{x_3}, \ldots \in \mathbb{R}^n$ with $\|\underline{x_k}\| = \frac{1}{k}$ (so $\lim_{k\to\infty} \|\underline{x_k}\| = 0$) and $\|\underline{x_k}\|_\infty = 1\ \forall k$. Now, it must be established that there is a convergent subsequence $\left(\underline{x_{k_p}}\right)_{p\geq 1} \subset (\underline{x_k})_{k\geq 1}$ such that $\left(\underline{x_{k_p}}\right) \to \underline{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$ as $p \to \infty$ with respect to $\|\cdot\|_\infty$. Let $\underline{x_k} = \left(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)}\right)$. Since $\|\underline{x_k}\|_\infty = 1$, $\left|x_k^{(1)}\right| \leq 1$, so $\left(x_k^{(1)}\right)_{k\geq 1}$ is bounded, so by the Bolzano-Weierstrass theorem $\exists \left(x_{k_p}^{(1)}\right)_{p\geq 1} \subset \left(x_k^{(1)}\right)_{k\geq 1}$ which converges to $x^{(1)} \in \mathbb{R}$. Applying this same argument to each component of $\underline{x_k}$ yields $\left(\underline{x_{k_p}}\right)_{p\geq 1}$ such that $\left(x_{k_p}^{(j)}\right)_{p\geq 1}$ converges to $x^{(j)} \in \mathbb{R}$ for any $1 \leq j \leq n$, so $\left(\underline{x_{k_p}}\right)_{p\geq 1} \to \underline{x}$ "component-wise" as $p \to \infty$. Since $\left(x_{k_p}^{(j)}\right)_{p\geq 1} \to x^{(j)}$, then, $\forall \varepsilon > 0$, $\exists N_\varepsilon(j)$ such that, $\forall n > N_\varepsilon(j)$, $\left|x_{k_p}^{(j)} - x^{(j)}\right| < \varepsilon$. Let $N_\varepsilon = \max_{1\leq j\leq n} N_\varepsilon(j)$. Then, $\forall k > N_\varepsilon$, $\left\|\underline{x_{k_p}} - \underline{x}\right\|_\infty = \max_{1\leq j\leq n} \left|x_{k_p}^{(j)} - x^{(j)}\right| < \varepsilon$, so $\lim_{p\to\infty} \left\|\underline{x_{k_p}} - \underline{x}\right\|_\infty = 0$, so $\left(\underline{x_{k_p}}\right)_{p\geq 1} \to \underline{x}$ as $p \to \infty$ with respect to $\|\cdot\|_\infty$.

Now, $1 = \left\|\underline{x_{k_p}}\right\|_\infty = \left\|\underline{x_{k_p}} - \underline{x} + \underline{x}\right\|_\infty \leq \left\|\underline{x_{k_p}} - \underline{x}\right\|_\infty + \|\underline{x}\|_\infty$, so $\|\underline{x}\|_\infty \geq 1 - \left\|\underline{x_{k_p}} - \underline{x}\right\|_\infty > 0$ since $\left\|x_{k_p} - \underline{x}\right\|_\infty \to 0$ as $p \to \infty$, so $\underline{x} \neq \underline{0}$ by the separation of points property of $\|\cdot\|_\infty$.

Also, $0 \leq \|\underline{x}\| = \left\|\underline{x} - \underline{x_{k_p}} + \underline{x_{k_p}}\right\| \leq \|x_{k_p}\| + \left\|\underline{x} - \underline{x_{k_p}}\right\| \leq \|x_{k_p}\| + k_2 \left\|\underline{x} - \underline{x_{k_p}}\right\|_\infty$. $(\underline{x_{k_p}}) \subset (\underline{x_k})$, so $\left\|\underline{x_{k_p}}\right\| \to 0$ as $p \to \infty$, and it was established that $\left\|\underline{x} - \underline{x_{k_p}}\right\|_\infty \to 0$ as $p \to \infty$, so $\|x_{k_p}\| + k_2 \left\|\underline{x} - \underline{x_{k_p}}\right\|_\infty \to 0$ as $p \to \infty$, so $\|\underline{x}\| = 0$, so $\underline{x} = 0$ by the separation of points property of $\|\cdot\|$.

However, this is a contradiction, so $J > 0$, so let $k_1 = J$. Then, from the definition of J, $k_1 \|\underline{x}\|_\infty \leq \|\underline{x}\|$ $\forall \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}$, and if $\underline{x} = \underline{0}$, then $\|\underline{x}\|_\infty = \|\underline{x}\| = 0$, so $k_1 \|\underline{x}\|_\infty = \|\underline{x}\|$, so $k_1 \|\underline{x}\|_\infty \leq \|\underline{x}\|$ $\forall \underline{x} \in \mathbb{R}^n$, so $k_1 \|\underline{x}\|_\infty \leq \|\underline{x}\| \leq k_2 \|\underline{x}\|_\infty$ $\forall \underline{x} \in \mathbb{R}^n$, so $\|\cdot\| \sim \|\cdot\|_\infty$ on $\mathbb{R}^n$, so all norms are equivalent on $\mathbb{R}^n$. ■

Proposition 37: The space $(\mathbb{R}^n, \|\cdot\|)$ is Banach for any norm $\|\cdot\|$ on $\mathbb{R}^n$.

Proof: It is enough to show that $(\mathbb{R}^n, \|\cdot\|_\infty)$ is complete, since if $\|\cdot\|_a \sim \|\cdot\|_b$, then $(V, \|\cdot\|_a)$ is Banach $\iff$ $(V, \|\cdot\|_b)$ is Banach, and $\|\cdot\|_\infty \sim \|\cdot\|$ for all norms $\|\cdot\|$ on $\mathbb{R}^n$ as shown previously.

Consider any Cauchy subsequence $(\underline{x_k})_{k\geq 1} \subset \mathbb{R}^n$. Then, for any $1 \leq j \leq n$, $0 \leq \left|x_{k+m}^{(j)} - x_k^{(j)}\right| \leq \max_{1\leq j\leq n}\left|x_{k+m}^{(j)} - x_k^{(j)}\right| = \left\|\underline{x_{k+m}} - \underline{x_k}\right\|_\infty \to 0$ as $k \to \infty$ since $(\underline{x_k})_{k\geq 1}$ is Cauchy in $\mathbb{R}^n$, so $\left(x_k^{(j)}\right)_{k\geq 1}$ is Cauchy in $\mathbb{R}$, so $\exists x^{(j)} \in \mathbb{R}$ such that $x_k^{(j)} \to x^{(j)}$ as $k \to \infty$. Let $\underline{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$. It was shown in the proof of the previous theorem that $\lim_{k\to\infty}\left\|\underline{x_k} - \underline{x}\right\|_\infty = 0$, so $(\underline{x_k})_{k\geq 1}$ converges in $\mathbb{R}^n$ with respect to $\|\cdot\|_\infty$, so every Cauchy sequence in $\mathbb{R}^n$ converges with respect to $\|\cdot\|_\infty$, so $(\mathbb{R}^n, \|\cdot\|_\infty)$ is complete, so $(\mathbb{R}^n, \|\cdot\|)$ is Banach for any norm $\|\cdot\|$ on $\mathbb{R}^n$. ■

Corollary: $(V, \|\cdot\|)$ is Banach for any norm $\|\cdot\|$ on V, provided $\dim V$ is finite. This is because any n-dimensional vector space V over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$, simply choose a basis in V.

Lemma (Cauchy-Schwarz Inequality for Integrals): $\forall f, g \in C[a,b]$,

$$\int_a^b fg \leq \|f\|_2 \|g\|_2 \text{, where } \|f\|_2 = \sqrt{\int_a^b f^2}$$

Proof: $\forall \lambda \in \mathbb{R}$,

$$\lambda^2 \int_a^b g^2 + 2\lambda \int_a^b fg + \int_a^b f^2 = \int_a^b (f + \lambda g)^2 \geq 0 \tag{1}$$

This is a polynomial in λ of degree 2 with at least one real root, so the discriminant

$$D = 4\left(\int_a^b fg\right)^2 - 4\|g\|_2^2\|f\|_2^2 \leq 0, \text{ so } \int_a^b fg \leq \|f\|_2\|g\|_2 \quad ■$$

Proposition 38:

1. The following functions are norms on $C[a,b]$:
   (a)
   $$\|f\|_\infty = \sup_{x\in[a,b]} \left|f(x)\right| \quad \text{(the sup norm)}$$
   (b)
   $$\|f\|_1 = \int_a^b |f| \quad \text{(a generalisation of the taxicab norm)}$$
   (c)
   $$\|f\|_2 = \sqrt{\int_a^b f^2} \quad \text{(a generalisation of the Euclidean norm)}$$
2. $(C[a,b], \|\cdot\|_\infty)$ is Banach, but $(C[a,b], \|\cdot\|_{1,2})$ is not Banach.
3. $\forall f \in C[0,1]$, $\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty$.
4. No pair of norms (a), (b) or (c) is equivalent.

Proof:

1. (a) It was demonstrated that $\|\cdot\|_\infty$ is a norm on $B[a,b]$, and $C[a,b] \subset B[a,b]$, so $\|\cdot\|_\infty$ is a norm on $C[a,b]$.

   (b) Separation of points: Assume that $\exists f$ such that $\|f\|_1 = 0$. Then $\int_a^b |f| = 0$. Assume that $f \not\equiv 0$. Then $\exists x_0 \in [a,b]$ such that $|f(x_0)| > 0$. But $f \in C[a,b]$, so $\exists [t_1, t_2] \subset [a,b]$ such that $\left| f\Big|_{[t_1,t_2]} \right| > 0$. Now, $0 = \|f\|_1 = \int_a^b |f| \geq \int_{t_1}^{t_2} |f| > 0$, so $0 > 0$, which is clearly a contradiction, so $f \equiv 0$.

   Absolute homogeneity: $\forall \lambda \in \mathbb{R}, \|\lambda \cdot f\|_1 = \int_a^b |\lambda \cdot f| = \int_a^b (|\lambda| \cdot |f|) = |\lambda| \cdot \int_a^b |f| = |\lambda| \cdot \|f\|_1$

   Triangle inequality: $\forall f, g \in C[a,b]$, $\|f+g\|_1 = \int_a^b |f+g| \leq \int_a^b (|f| + |g|) = \int_a^b |f| + \int_a^b |g| = \|f\|_1 + \|g\|_1$

   (c) The proofs of the first two properties are very similar to those in (b), and so they are left as exercises.

   Triangle inequality: Using the Cauchy-Schwarz inequality

$$\|f+g\|_2^2 = \int_a^b (f+g)^2 = \int_a^b f(f+g) + \int_a^b g(f+g) \leq \|f\|_2 \|f+g\|_2 + \|g\|_2 \|f+g\|_2$$

   Then, assuming that $(f+g) \not\equiv 0$,

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$$

   If $f + g \equiv 0$ then the proof is trivial.

2. Consider the sequence of functions $(f_k)_{k \geq 1} \subset C[a,b]$ given by

$$f_k(x) = \begin{cases} -1 & a \leq x < \frac{a+b}{2} - \frac{1}{k} \\ k(x - \frac{a+b}{2}) & \frac{a+b}{2} - \frac{1}{k} \leq x \leq \frac{a+b}{2} + \frac{1}{k} \\ 1 & \frac{a+b}{2} + \frac{1}{k} < x \leq b \end{cases}$$

   Then, for any $k, m \in \mathbb{N}$,

$$\|f_k - f_{k+m}\|_1 = \int_a^b |f_k - f_{k+m}| = \int_{\frac{a+b}{2} - \frac{1}{k}}^{\frac{a+b}{2} + \frac{1}{k}} |f_k - f_{k+m}|$$

   since the functions are equal outside of these bounds. Now, f_k is bounded by $[-1,1]$, so $|f_k - f_{k+m}| \leq 2$, so

$$\|f_k - f_{k+m}\|_1 \leq \int_{\frac{a+b}{2} - \frac{1}{k}}^{\frac{a+b}{2} + \frac{1}{k}} 2 = 2 \cdot \frac{2}{k} = \frac{4}{k} \to 0 \text{ as } k \to \infty$$

   so $(f_k)_{k \geq 1}$ is Cauchy with respect to $\|\cdot\|_1$. Suppose that it has a limit $f \in C[a,b]$. Then, $\forall \varepsilon > 0$, $f\left(\frac{a+b}{2} - \varepsilon\right) = -1$ and $f\left(\frac{a+b}{2} + \varepsilon\right) = 1$, since otherwise, for any $k > \frac{1}{\varepsilon}$, $\int_a^b |f_k - f| > 0$ which contradicts f being the limit of $(f_k)_{k \geq 1}$. But then f is not continuous, since

$$-1 = \lim_{x \uparrow \frac{a+b}{2}} f(x) \neq \lim_{x \downarrow \frac{a+b}{2}} f(x) = 1$$

   This is a contradiction, so $(f_k)_{k \geq 1}$ does not have a limit in $C[a,b]$ with respect to $\|\cdot\|_1$, but it is Cauchy with respect to $\|\cdot\|_1$, so $(C[a,b, \|\cdot\|_1)$ is not Banach.

3. For any $f \in C[0,1]$, and some $x_0 \in [0,1]$

$$\|f\|_1 = \int_0^1 |f| \leq \int_0^1 \|f\|_\infty = \|f\|_\infty$$

$$\|f\|_1 = \int_0^1 |f| \cdot 1 \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2$$

$$\|f\|_2 = \left(\int_0^1 f^2 \right)^{\frac{1}{2}} \leq \left(\left\| f^2 \right\|_\infty \right)^{\frac{1}{2}} = \left(\sup_{x \in [0,1]} f(x)^2 \right)^{\frac{1}{2}} = \left(f(x_0)^2 \right)^{\frac{1}{2}} = |f(x_0)| = \|f\|_\infty$$

4. This is mostly left as an exercise, however a proof that $\|\cdot\|_1 \not\sim \|\cdot\|_2$ will be given. For simplicity, this shall be proven with respect to $C[0,1]$, the generalisation is left as an easy exercise.

   Let, for any $\varepsilon > 0$,

$$f_\varepsilon(x) = \begin{cases} \frac{1}{\sqrt{\varepsilon}} & 0 \leq x < \varepsilon \\ \frac{1}{\sqrt{x}} & \varepsilon \leq x \leq 1 \end{cases} \in C[0,1]$$

   Then

$$\|f_\varepsilon\|_1 = \int_0^1 |f_\varepsilon| = \int_0^\varepsilon \frac{1}{\sqrt{\varepsilon}}\, dx + \int_\varepsilon^1 \frac{1}{\sqrt{x}}\, dx = \left[\sqrt{\varepsilon} + 2\sqrt{x}\right]_1^\varepsilon = 2 - \sqrt{\varepsilon} \leq 2$$

$$\|f_\varepsilon\|_2^2 = \int_0^1 f_\varepsilon^2 = \int_0^\varepsilon \frac{1}{\varepsilon}\, dx + \int_\varepsilon^1 \frac{1}{x}\, dx = [1 + \log x]_\varepsilon^1 = 1 + \log\frac{1}{\varepsilon} \to \infty \text{ as } \varepsilon \downarrow 0$$

   So $\nexists k > 0$ such that $\|f\|_2 \leq k\|f\|_1 \ \forall f \in C[0,1]$. ■

Exercise: Using a proof similar to the one in 3, show that $(C[a,b], \|\cdot\|_2)$ is not Banach.

10 Continuous Maps

10.1 Definition & Basic Properties of Linear Maps

Definition 14: Let $f : (V,\|\cdot\|_V) \to (W,\|\cdot\|_W)$ be a map from V to W. Then f is called $(\|\cdot\|_V, \|\cdot\|_W)$ *continuous* (or simply *continuous* when the spaces in question are clear) at $\underline{u} \in V$ if, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that, $\forall \underline{v} \in V, \|\underline{u} - \underline{v}\|_V < \delta_\varepsilon \implies \|f(\underline{u}) - f(\underline{v})\|_W < \varepsilon$.

Example: Let $I : (C[a,b],\|\cdot\|_\infty) \to (\mathbb{R},|\cdot|)$ be given by

$$I : f \in C[a,b] \mapsto \int_a^b f \in \mathbb{R}$$

It was demonstrated in the first section that this map is linear.

Theorem 39: Let $T : (V,\|\cdot\|_V) \to (W,\|\cdot\|_W)$ be a linear map. Then the following conditions are equivalent:

(i) T is continuous at $\underline{0_V} \in V$.

(ii) T is continuous on V (T is continuous at every point of V).

(iii) The set $\left\{\|T(\underline{x})\|_W : \underline{x} \in V \text{ such that } \|\underline{x}\|_V \leq 1\right\} \subset \mathbb{R}$ is bounded (then T is called *bounded*).

Proof (i) $\iff$ (ii): The $\impliedby$ direction is clear, if T is continuous on V, it is continuous at $\underline{0_V} \in V$.

For the $\implies$ direction, suppose that T is continuous at $\underline{0_V}$. $T(\underline{0_V}) = \underline{0_W}$, so, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $\|\underline{x}\|_V < \delta_\varepsilon \implies \|T(\underline{x})\|_W < \varepsilon$. Choose some $\underline{u} \in V$. Then, for any $\underline{v} \in V$, let $\underline{x} = \underline{u} - \underline{v}$. Now, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $\|\underline{u} - \underline{v}\|_V < \delta_\varepsilon \implies \|T(\underline{u}) - T(\underline{v})\|_W = \|T(\underline{u} - \underline{v})\|_W < \varepsilon$, so T is continuous at $\underline{u}$. Since the choice of $\underline{u}$ was arbitrary, T is continuous on V.

Proof (i) $\iff$ (iii): For the $\implies$ direction, by continuity, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that, $\forall x \in V$, $\|\underline{x}\|_V \leq \tilde{\delta_\varepsilon} = \frac{\delta_\varepsilon}{2} \implies \|T(\underline{x})\|_W < \varepsilon$. Then $\left\|\frac{\underline{x}}{\delta_\varepsilon}\right\|_V \leq 1 \implies \left\|T\left(\frac{\underline{x}}{\delta_\varepsilon}\right)\right\|_W \leq \frac{\varepsilon}{\delta_\varepsilon}$, so $\forall \underline{x} \in V, \|\underline{x}\|_V \leq 1 \implies \|T(\underline{x})\|_W < \frac{\varepsilon}{\delta_\varepsilon}$, so $\left\{\|T(\underline{x})\|_W : \underline{x} \in V \text{ such that } \|\underline{x}\|_V \leq 1\right\}$ is bounded by $\frac{\varepsilon}{\delta_\varepsilon}$, so it is bounded.

For the $\impliedby$ direction, T is bounded, so $\exists B > 0$ such that $\forall \underline{u}$, $\|\underline{u}\|_V \leq 1 \implies \|T(\underline{u})\|_W < B$. Then, for any $\varepsilon > 0$, multiplying this inequality by $\frac{\varepsilon}{B}$ yields $\left\|\frac{\varepsilon \underline{u}}{B}\right\|_V \leq \frac{\varepsilon}{B} \implies \left\|T\left(\frac{\varepsilon \underline{u}}{B}\right)\right\|_W < \varepsilon$, so $\forall \underline{v} \in V$, $\|\underline{v}\|_V < \delta_\varepsilon = \frac{\varepsilon}{B} \implies \|T(\underline{v})\|_W < \varepsilon$, so T is continuous at $\underline{0_V}$ with $\delta_\varepsilon = \frac{\varepsilon}{B}$. ■

Example: For any $f \in C[a,b]$, $\|f\|_\infty \leq 1$,

$$\int_a^b \leq \|f\|_\infty \cdot |b - a| \leq |b - a|$$

so $I : C[a,b] \to \mathbb{R}$,

$$I : f \in C[a,b] \mapsto \int_a^b f \in \mathbb{R}$$

is bounded, so I is continuous on $(C[a,b],\|\cdot\|_\infty)$.

Proposition 40: Let V and W be vector spaces, with norms $\|\cdot\|_{V_1} \sim \|\cdot\|_{V_2}$ and $\|\cdot\|_{W_1} \sim \|\cdot\|_{W_2}$ respectively, and let $T : V \to W$ be a linear map. Then

(i) T is $(\|\cdot\|_{V_1}, \|\cdot\|_{W_1})$ continuous iff T is $(\|\cdot\|_{V_2}, \|\cdot\|_{W_2})$ continuous.

(ii) Any linear map $(\mathbb{R}^n, \|\cdot\|_1) \to (W, \|\cdot\|_W)$ is continuous.

(iii) Any linear map $(\mathbb{R}^n, \|\cdot\|) \to (W, \|\cdot\|_W)$ is continuous, where $\|\cdot\|$ is an arbitrary norm.

(iv) Let $F : (C[0,1], \|\cdot\|_{1,\infty}) \to (\mathbb{R}, |\cdot|)$ be an evaluation map with $F : f \mapsto f(0) \in \mathbb{R}$. Then F is $(\|\cdot\|_\infty, |\cdot|)$ continuous but not $(\|\cdot\|_1, |\cdot|)$ continuous.

Proof:

(i) T is $(\|\cdot\|_{V_1}, \|\cdot\|_{W_1})$ continuous, so T is bounded with respect to $(\|\cdot\|_{V_1}, \|\cdot\|_{W_1})$, so $\exists B \geq 0$ such that $\forall u \in V$, $\|\underline{u}\|_{V_1} \leq 1 \implies \|T(\underline{u})\|_{W_1} \leq B$. Since $\|\cdot\|_{V_1} \sim \|\cdot\|_{V_2}$, $\exists K > 0$ such that, $\forall \underline{u} \in V$, $\|\underline{u}\|_{V_1} \leq K\|\underline{u}\|_{V_2}$. Consider all $\underline{u} \in V$ such that $K\|\underline{u}\|_{V_2} \leq 1$. Then $\|\underline{u}\|_{V_1} \leq 1$, so $\|T(\underline{u})\|_{W_1} \leq B$. Now, $\|\cdot\|_{W_1} \sim \|\cdot\|_{W_2}$, so $\exists L > 0$ such that, $\forall \underline{v} \in W$, $L\|\underline{v}\|_{W_2} \leq \|\underline{v}\|_{W_1}$, so $L\|T(\underline{u})\|_{W_2} \leq \|T(\underline{u})\|_{W_1} \leq B$. Then, $K\|\underline{u}\|_{V_2} \leq 1 \implies L\|T(\underline{u})\|_{W_2} \leq B$. Let $\underline{w} = K\underline{u}$. Then, $\forall \underline{w} \in V$, $\|\underline{w}\|_{V_2} \leq 1 \implies \|T(\underline{w})\|_{W_2} \leq \frac{KB}{L} = B'$, so T is bounded with respect to $(\|\cdot\|_{V_2}, \|\cdot\|_{W_2})$, so T is $(\|\cdot\|_{V_2}, \|\cdot\|_{W_2})$ continuous.

(ii) For any $\underline{x} \in \mathbb{R}^n$, $\exists x_1, \dots, x_n \in \mathbb{R}$ such that, for some basis $\underline{e_1}, \dots, \underline{e_n} \in \mathbb{R}^n$,

$$\underline{x} = \sum_{i=1}^n x_i \underline{e_i}, \text{ so } \|\underline{x}\|_1 = \sum_{i=1}^n |x_i|$$

Now, consider any $\underline{x} \in \mathbb{R}^n$ with $\|\underline{x}\|_1 \leq 1$. Then

$$\begin{aligned}\|T(\underline{x})\|_W &= \left\| T\left(\sum_{i=1}^n x_i \underline{e_i}\right) \right\|_W = \left\| \sum_{i=1}^n x_i T(\underline{e_i}) \right\|_W \\ &\leq \sum_{i=1}^n |x_i| \|T(\underline{e_i})\|_W \leq \max_{1 \leq i \leq n} |T(\underline{e_i})| \sum_{i=1}^n |x_i| \leq \max_{1 \leq i \leq n} |T(\underline{e_i})| := B\end{aligned}$$

So $\exists B > 0$ such that, $\forall \underline{x} \in \mathbb{R}^n$ with $\|\underline{x}\|_1 \leq 1$, $\|T(\underline{x})\|_W \leq B$, so T is bounded, so T is continuous.

(iii) This is a direct consequence of (i) and (ii), since all norms on $\mathbb{R}^n$ are equivalent.

(iv) For $F : (C[0,1], \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$, take any $f \in C[0,1]$ such that $\|f\|_\infty \leq 1$. Then $|f(0)| \leq 1$, so $|F(f)| \leq 1$, so F is bounded by 1, so F is continuous.

For $F : (C[0,1], \|\cdot\|_1) \to (\mathbb{R}, |\cdot|)$, let $(f_n)_{n \geq 1} \subset C[a,b]$ be the sequence given by $f_n : [0,1] \to \mathbb{R}$, $f_n : x \mapsto (\frac{1}{n} - n)x + n$. Then

$$\|f_n\|_1 = \int_0^1 f_n = \int_0^1 \left(\frac{1}{n} - n\right) x + n \, dx = \frac{1}{n} \leq 1 \; \forall n \in \mathbb{N}$$

However, $|F(f_n)| = |f_n(0)| = n \to \infty$ as $n \to \infty$, so $\left\{ |F(f)| : \|f\|_1 \leq 1 \right\}$ is not bounded, so F is not continuous. ■

10.2 Spaces of Bounded Linear Maps

Theorem 41: Suppose $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$ are normed spaces, let

$$L(V,W) = \{T : V \to W : T \text{ is linear and continuous/bounded}\}$$

and let

$$\|\cdot\| : T \in L(V,W) \mapsto \sup_{\underline{x}\in V:\|\underline{x}\|_V\leq 1} \left\|T(\underline{x})\right\|_W$$

Then $(L(V,W),\|\cdot\|)$ is a normed vector space.

Proof: For any $\lambda,\mu \in \mathbb{R}$, $T,G \in L(V,W)$, $\underline{x} \in V$, $(\lambda T + \mu G)(\underline{x}) = \lambda T(\underline{x}) + \mu G(\underline{x})$ - a map from V to W. Take any $\alpha,\beta \in \mathbb{R}$, $\underline{u},\underline{v} \in V$. Then $(\lambda T + \mu G)(\alpha\underline{u} + \beta\underline{v}) = \lambda T(\alpha\underline{u} + \beta\underline{v}) + \mu G(\alpha\underline{u} + \beta\underline{v}) = \lambda(\alpha T(\underline{u}) + \beta T(\underline{v})) + \mu(\alpha G(\underline{u}) + \beta G(\underline{v})) = \alpha(\lambda T + \mu G)(\underline{u}) + \beta(\lambda T + \mu G)(\underline{v})$, so $(\lambda T + \mu G)$ is linear. Now, $\|\lambda T + \mu G\| = \sup_{\|\underline{x}\|_V\leq 1}\left\|\lambda T(\underline{x}) + \mu G(\underline{x})\right\|_W \leq |\lambda|\sup_{\|\underline{x}\|_V\leq 1}\left\|T(\underline{x})\right\|_W + |\mu|\sup_{\|\underline{x}\|_V\leq 1}\left\|G(\underline{x})\right\|_W = |\lambda|\|T\| + |\mu|\|G\|$. Then, since T and G are both bounded, $\lambda T + \mu G$ is bounded, so $\lambda T + \mu G$ is continuous, so $L(V,W)$ is closed under linear combinations of maps. Verification of the remaining axioms of a vector space is left as an exercise.

It remains to show that $\|\cdot\|$ is a norm. Taking $\lambda = \mu = 1$ in the previous section of the proof yields $\|T + G\| \leq \|T\| + \|G\|$ which proves the triangle inequality. The proof of absolute homogeneity is left as an easy exercise. To prove separation of points, suppose that $\|T\| = 0_{L(V,W)}$. Then $\sup_{\|\underline{u}\|_V\leq 1}\left\|T(\underline{u})\right\|_W = 0_{L(V,W)}$, so $\left\|T(\underline{u})\right\|_W = 0_{L(V,W)}$ $\forall \underline{u} \in V$ such that $\|\underline{u}\|_V \leq 1$. Since $\|\cdot\|_W$ is a norm, $T(\underline{x}) = 0_{L(V,W)}$ $\forall \underline{u} \in V$ such that $\|\underline{u}\|_V \leq 1$. Now, take any $\underline{v} \in V$ such that $\|\underline{v}\|_V > 1$. Then $\left\|T(\underline{v})\right\|_W = \|\underline{v}\|_V \cdot \left\|T\left(\frac{\underline{v}}{\|\underline{v}\|_V}\right)\right\|_W = 0_{L(V,W)}$, so $T(\underline{x}) = 0_{L(V,W)}$ $\forall \underline{x} \in V$, so $T = 0_{L(V,W)}$, which completes the proof. ■

Remarks:

1. The norm $\|\cdot\|$ is called an *operator norm.*
2. Consider $L(\mathbb{R}^m,\mathbb{R}^n)$ - this is the space $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices.
3. Suppose then that $m = n$ and that both of these $\mathbb{R}^n$s are equipped with the $\|\cdot\|_2$ norm. Then for any $T \in \mathbb{R}^{n\times n}$, $\|T\| = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigenvalue of TT', with T' being the transpose of T.
4. If $(W,\|\cdot\|_W)$ is Banach, then $(L(V,W),\|\cdot\|)$ is Banach also.

Exercise: Prove 4 and that $\|T\| = \sqrt{\lambda_{\max}}$ as given in 3 is a norm.

11 Open and Closed Sets of Normed Spaces

11.1 Definition & Basic Properties of Open Sets

Definition 15: Let $(V,\|\cdot\|)$ be a normed space, and let $\underline{u} \in V$, $\delta > 0$. Then the set

$$B(\underline{u}, \delta) = \{\underline{v} \in V : \|\underline{v} - \underline{u}\| < \delta\} \subset (V,\|\cdot\|)$$

is called an *open ball of radius δ centred at $\underline{u}$*. $B(\underline{u}, \delta,\|\cdot\|)$ denotes an open ball defined with respect to $\|\cdot\|$, $B(\underline{u}, \delta)$ is shorthand when the norm in question is obvious.

Examples:

1. For $(\mathbb{R},|\cdot|)$, $B(x, \delta) = (x - \delta, x + \delta)$.
2. For $(\mathbb{R}^2,\|\cdot\|_1)$, $B_1(0, 1) = \{(\underline{x_1}, \underline{x_2} \in \mathbb{R}^2 : |\underline{x_1}| + |\underline{x_2}| < 1)\}$.
   For $(\mathbb{R}^2,\|\cdot\|_2)$, $B_2(0, 1) = \{(\underline{x_1}, \underline{x_2} \in \mathbb{R}^2 : \sqrt{\underline{x_1}^2 + \underline{x_2}^2} < 1)\}$.
   For $(\mathbb{R}^2,\|\cdot\|_\infty)$, $B_\infty(0, 1) = \{(\underline{x_1}, \underline{x_2} \in \mathbb{R}^2 : \max(\underline{x_1}, \underline{x_2}) < 1)\}$.
3. For $(C[a, b],\|\cdot\|_\infty)$, $B(g, \varepsilon) = \{f \in C[a, b] : \sup_{x\in[a,b]}|f(x) - g(x)| < \varepsilon\}$

Definition 15.1: Let $(V,\|\cdot\|)$ be a normed space. A subset $U \subset V$ is called *open* if, $\forall \underline{x} \in V$, $\exists \delta_{\underline{x}}$ such that $B(\underline{x}, \delta_{\underline{x}}) \subset U$.

Exercise: Prove that the empty set is open.

Lemma 42: Let $(V,\|\cdot\|)$ be a normed space. Then

(i) Open balls in V are open sets.

(ii) $\forall \underline{u} \in V, \delta > 0$, $B(\underline{u}, \delta) = \underline{u} + \delta B(0, 1) := \{\underline{v} \in V : \underline{v} = \underline{u} + \delta\underline{w}, \underline{w} \in B(0, 1)\}$.

Proof:

(i) $\underline{v} \in B(\underline{u}, \delta) \iff \|\underline{v} - \underline{u}\| < \delta$. Let $\delta' = \delta - \|\underline{v} - \underline{u}\| > 0$. Consider $B(\underline{v}, \delta')$. Then, for any $\underline{w} \in B(\underline{v}, \delta')$, $\|\underline{w} - \underline{v}\| < \delta'$. Then $\|\underline{u} - \underline{w}\| = \|\underline{u} - \underline{v} + \underline{v} - \underline{w}\| \leq \|\underline{u} - \underline{v}\| + \|\underline{v} - \underline{w}\| < \|\underline{u} - \underline{v}\| + \delta' = \|\underline{u} - \underline{v}\| + \delta - \|\underline{v} - \underline{u}\| = \delta$, so $B(\underline{v}, \delta') \subset B(\underline{u}, \delta)$. Then, since the choice of $\underline{v}$ was arbitrary, $B(\underline{u}, \delta)$ is open.

(ii) Take $\underline{v} \in B(\underline{u}, \delta)$, then $\|\underline{u} - \underline{v}\| < \delta$, so $\left\|\frac{\underline{u}-\underline{v}}{\delta}\right\| < 1$. Let $\underline{w} = \frac{\underline{u}-\underline{v}}{\delta}$. Then $\underline{w} \in B(0, 1)$, so $\underline{v} = \underline{u} + \delta\underline{w}$ with $\underline{w} \in B(0, 1)$, so $\underline{v} \in \underline{u} + \delta B(0, 1)$, so $B(\underline{u}, \delta) \subseteq \underline{u} + \delta B(0, 1)$.

Now, take any $\underline{v} \in \underline{u} + \delta B(0, 1)$. Then $\exists \underline{w} \in B(0, 1)$ such that $\underline{v} = \underline{u} + \delta\underline{w}$, so $\|\underline{u} - \underline{v}\| = \|\delta\underline{w}\| = \delta\|\underline{w}\| < \delta$, so $\underline{v} \in B(\underline{u}, \delta)$. Then $\underline{u} + \delta B(0, 1) \subseteq B(\underline{u}, \delta)$, so $B(\underline{u}, \delta) = \underline{u} + \delta B(0, 1)$. ∎

Lemma 43: Let $\|\cdot\|_A \sim \|\cdot\|_B$ be equivalent norms on V. Then $U \subset V$ is open with respect to $\|\cdot\|_A$ iff $U \subset V$ is open with respect to $\|\cdot\|_B$.

Proof: Assume without loss of generality that V is open with respect to $\|\cdot\|_A$. Then, $\forall \underline{u} \in U$, $\exists \delta_{\underline{u}} > 0$ such that $B(\underline{u}, \delta_{\underline{u}},\|\cdot\|_A) \subset U$. Now, consider some $\delta'_{\underline{u}} > 0$. Then $\underline{v} \in B(\underline{u}, \delta'_{\underline{u}},\|\cdot\|_B) \iff \|\underline{v} - \underline{u}\|_B < \delta'_{\underline{u}}$. Since $\|\cdot\|_A \sim \|\cdot\|_B$, $\exists K > 0$ such that $\|\underline{v} - \underline{u}\|_A < K\|\underline{v} - \underline{u}\|_B < K\delta'_{\underline{u}}$. Let $\delta'_{\underline{u}} = \frac{\delta_{\underline{u}}}{K}$. Then $B(\underline{u}, \delta'_{\underline{u}},\|\cdot\|_B) \subset B(\underline{u}, \delta_{\underline{u}},\|\cdot\|_A)$, so V is open with respect to $\|\cdot\|_B$. ∎

Example: If $\underline{v} \in \mathbb{R}^n$ is open with respect to $\|\cdot\|_1$, then V is open with respect to an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$ since all norms on $\mathbb{R}^n$ are equivalent.

11.2 Continuity, Closed Sets & Convergence

Definition 15.2: A function $f : (V,\|\cdot\|_V) \to (W,\|\cdot\|_W)$ is *continuous at* $\underline{x} \in V$ if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $f(B(\underline{x}, \delta,\|\cdot\|_V)) \subset B(f(\underline{x}), \epsilon,\|\cdot\|_W)$.

Exercise: Prove that this definition is equivalent to the previous definition of continuity.

Definition 15.3: A set $U \subset (V,\|\cdot\|)$ is *bounded* if it is contained in a ball of finite radius. Equivalently, $U \subset V$ is bounded if $\exists \underline{x} \in V, \delta > 0$ such that $U \subset B(\underline{x}, \delta)$.

Proposition 44: Let $f : (V,\|\cdot\|_V) \to (W,\|\cdot\|_W)$. Then f is continuous on V iff the preimage of any open subset $U \subset W$ is open. Equivalently, $f : V \to W$ is continuous iff, $\forall U \subset W$ such that U is open, $f^{-1}(U) := \{\underline{x} \in V : f(\underline{x}) \in U\}$ is open.

Proof $\implies$: Let $U \subset W$ be an open subset. Then, either $f^{-1}(U) = \varnothing$ which is open, or $f^{-1}(U) \neq \varnothing$. If $f^{-1}(U) \neq \varnothing$, then let $\underline{u} \in f^{-1}(U) \subset V$ be any point in the preimage. Let $\underline{v} = f(\underline{u}) \in U$. Then $\exists \varepsilon > 0$ such that $B(\underline{v}, \varepsilon,\|\cdot\|_W) \subset U$. But f is continuous on V, so f is continuous at $\underline{u}$, so $\exists \delta_\varepsilon > 0$ such that $f(B(\underline{u}, \delta_\varepsilon,\|\cdot\|_V)) \subset B(\underline{v}, \varepsilon,\|\cdot\|_W) \subset U$, so $B(\underline{u}, \delta_\varepsilon,\|\cdot\|_V) \subset f^{-1}(U)$, so $f^{-1}(U)$ is open.

Proof $\impliedby$: Take any $\underline{u} \in V$, and let $\underline{v} = f(\underline{u})$. For any $\varepsilon > 0$, consider $B(\underline{v}, \varepsilon,\|\cdot\|_W)$. $f^{-1}(B(\underline{v}, \varepsilon,\|\cdot\|_W))$ is open, and $\underline{u} \in f^{-1}(B(\underline{v}, \varepsilon,\|\cdot\|_W))$. Since $f^{-1}(B(\underline{v}, \varepsilon,\|\cdot\|_W))$ is open, $\exists \delta_\varepsilon$ such that $B(\underline{u}, \delta_\varepsilon,\|\cdot\|_V) \subset f^{-1}(B(\underline{v}, \varepsilon,\|\cdot\|_W))$, so $f(B(\underline{u}, \delta_\varepsilon,\|\cdot\|_V)) \subset (B(\underline{v}, \varepsilon,\|\cdot\|_W)$, so f is continuous on V. ■

Definition 16: A set $U \subset (V,\|\cdot\|)$ is called *closed* if $V \setminus U$ is open.

Examples:

(i) $[a, b] \subset \mathbb{R}$ is closed since $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open.

(ii) $[a, b) \subset \mathbb{R}$ is neither open nor closed (it is not open since $B(a, \varepsilon) \not\subset [a, b)$ for any $\varepsilon > 0$, and it is not closed since $\mathbb{R} \setminus [a, b) = (-\infty, a) \cup [b, \infty)$ is not open).

Theorem 45: Let $(V,\|\cdot\|)$ be a normed space. Then $U \subset V$ is closed iff, for any sequence $(\underline{x_n}) \subset U$ which converges to $\underline{x} \in V$, $\underline{x} \in U$.

Proof $\implies$: Let $U \subset V$ be a closed subset. Assume that $U \neq \varnothing$, the case when $U = \varnothing$ is trivial. Then let $(\underline{x_n}) \subset U$ be a convergent sequence, with $\underline{x} = \lim_{n\to\infty} \underline{x_n}$, so $\lim_{n\to\infty}\|\underline{x} - \underline{x_n}\| = 0$. Suppose that $\underline{x} \notin U$, then $\underline{x} \in V \setminus U$. Since U is closed, $V \setminus U$ is open, so $\exists \delta > 0$ such that $B(\underline{x}, \delta) \subset V \setminus U$. However, since $\underline{x_n} \to \underline{x}$ as $n \to \infty$, $\exists N_\delta$ such that, $\forall n > N_\delta$, $\|\underline{x} - \underline{x_n}\| < \delta$, so $\underline{x_n} \in B(\underline{x}, \delta) \subset V \setminus U$, which is a contradiction since $\underline{x_n} \in U$, so $\underline{x} \in U$.

Proof $\impliedby$: Now, any sequence in U which converges does so to a point in U. Suppose that U is not closed. Then $V \setminus U$ is not open, so $\exists \underline{x} \in V \setminus U$ such that, $\forall \varepsilon > 0$, $B(\underline{x}, \varepsilon) \not\subset V \setminus U$, so $B(\underline{x}, \varepsilon) \cap U \neq \varnothing$. Let (ε_n) be any null sequence. Then, for any $n \in \mathbb{N}$, $B(\underline{x}, \varepsilon_n) \cap U \neq \varnothing$, so $\exists \underline{x_n} \in U$ such that $\|\underline{x} - \underline{x_n}\| < \varepsilon_n$, so $\lim_{n\to\infty}\|\underline{x} - \underline{x_n}\| = 0$. Then $(\underline{x_n}) \subset U$ is a sequence which converges to a point $\underline{x} \notin U$, which is a contradiction, so U is closed. ■

Notes: Consider the normed space $(V,\|\cdot\|)$. For any $\underline{x} \in V$ and any $\delta > 0$, $B(\underline{x}, \delta) \subset V$, so V is open. However, any sequence in V which converges does so in V, so V is closed also. $\varnothing$ is also both closed and open, since $\varnothing = V \setminus V$. These sets are called *clopen*.

11.3 Lipschitz Continuity, Contraction Mappings & Other Remarks

Remarks: Let $(V,\|\cdot\|)$ be a normed space. Then

1. The map $\|\cdot\| : (V,\|\cdot\|) \to (\mathbb{R},|\cdot|)$ is continuous. To see this, fix any $\underline{u} \in V$. Then, for any $\underline{v} \in V$, $\big\|\|\underline{u}\| - \|\underline{v}\|\big\| \le \|\underline{u} - \underline{v}\|$ by the triangle inequality. Now, if $\|\underline{u} - \underline{v}\| < \varepsilon$, then $\big\|\|\underline{u}\| - \|\underline{v}\|\big\| < \varepsilon$, so $\big\|B(\underline{u}, \varepsilon, \|\cdot\|)\big\| \subset B\left(\|\underline{u}\|, \varepsilon, |\cdot|\right)$, so setting $\delta_\varepsilon = \varepsilon$ demonstrates that $\|\cdot\| : V \to \mathbb{R}$ is continuous.

2. Suppose that, for a map $f : (V,\|\cdot\|) \to (V,\|\cdot\|)$, $\exists K \in (0,1)$ such that, $\forall \underline{u}, \underline{v} \in V$, $\big\|f(\underline{u}) - f(\underline{v})\big\| \le K\|\underline{u} - \underline{v}\|$. Then f is continuous on V. To see this, note that $f(B(\underline{v}, \frac{\varepsilon}{2K})) \subset B(f(\underline{v}), \varepsilon)$, the details of the proof are left as an exercise. Such a map is called a *contraction mapping*. In fact, if this condition holds for any $k > 0$, continuity is preserved, and the map is called *Lipschitz continuous*.

3. A *closed ball of radius δ centred at* $\underline{u}$ is defined as $\overline{B}(\underline{u}, \delta) \coloneqq \{\underline{v} \in V : \|\underline{u} - \underline{v}\| \le \delta\} \subset (V,\|\cdot\|)$. This set is closed. To see this, let $(\underline{u_n})_{n\ge1} \subset \overline{B}(\underline{u}, \delta)$ such that $\underline{u_n} \to \underline{w} \in V$ as $n \to \infty$. Then, by the continuity of $\|\cdot\|$ at $\underline{u} \in V$, $\|\underline{w} - \underline{u}\| = \lim_{n\to\infty} \big\|\underline{u_n} - \underline{u}\big\| \le \delta$, so $\underline{w} \in \overline{B}(\underline{u}, \delta)$, so $\overline{B}(\underline{u}, \delta)$ is closed.

4. Let $U \subset (V,\|\cdot\|)$. Then $\underline{w} \in V$ is a *boundary point* of U if, $\forall \varepsilon > 0$, $B(\underline{w}, \varepsilon) \cap U \ne \varnothing$ and $B(\underline{w}, \varepsilon) \cap V \setminus U \ne \varnothing$. $\underline{w} \in V$ is a *limit point* of U if $\exists (\underline{x_n})_{n\ge1} \subset U$ which converges to $\underline{w}$.

12 The Contraction Mapping Theorem & Applications

12.1 Statement and Proof

Theorem 46 - The Contraction Mapping/Banach Fixed Point Theorem: Let $(V, \|\cdot\|)$ be a Banach space, $U \subset V$ a closed subset of this space, and $f : U \to U$ a contraction mapping. Then $\exists! \underline{w} \in U$ such that $f(\underline{w}) = \underline{w}$. Moreover, for any $\underline{u} \in U$, the sequence $(\underline{u_n})_{n\geq 0} \subset U$ converges to $\underline{w}$ as $n \to \infty$, where $\underline{u_0} = \underline{u}$ and $\underline{u_{n+1}} = f(\underline{u_n})$.

Proof: f is a contraction mapping, so $\exists K \in (0,1)$ such that, $\forall \underline{u}, \underline{v} \in U$, $\|f(\underline{u}) - f(\underline{v})\| \leq K\|\underline{u} - \underline{v}\|$. Then $\left\|\underline{u_{n+1}} - \underline{u_n}\right\| = \left\|f(\underline{u_n}) - f(\underline{u_{n-1}})\right\| \leq K\left\|\underline{u_n} - \underline{u_{n-1}}\right\| \leq K^2\left\|\underline{u_{n-1}} - \underline{u_{n-2}}\right\| \leq \ldots \leq K^n\|\underline{u_1} - \underline{u_0}\|$. Now,

$$\begin{aligned}
\left\|\underline{u_{n+m}} - \underline{u_n}\right\| &= \left\|\underline{u_{n+m}} - \underline{u_{n+m-1}} + \underline{u_{n+m-1}} - \underline{u_{n+m-2}} + \underline{u_{n+m-2}} - \ldots + \underline{u_{n+1}} - \underline{u_n}\right\| \\
&\leq \left\|\underline{u_{n+m}} - \underline{u_{n+m-1}}\right\| + \ldots + \left\|\underline{u_{n+1}} - \underline{u_n}\right\| \\
&\leq (K^{n+m-1} + K^{n+m-2} + \ldots + K^n)\|\underline{u_1} - \underline{u_0}\| \\
&\leq \sum_{j=0}^{\infty} K^{n+j}\|\underline{u_1} - \underline{u_0}\| \\
&= \frac{K^n}{1-K}\|\underline{u_1} - \underline{u_0}\| \to 0 \text{ as } n \to \infty,
\end{aligned}$$

so $(\underline{u_n})_{n\geq 0}$ is Cauchy. Since $(V, \|\cdot\|)$ is Banach and U is closed, $(\underline{u_n}) \to \underline{w} \in U$ as $n \to \infty$. Now, consider $\underline{u_{n+1}} = f(\underline{u_n})$. Then $\lim_{n\to\infty} \underline{u_{n+1}} = \lim_{n\to\infty} f(\underline{u_n})$. $\lim_{n\to\infty} \underline{u_{n+1}} = \underline{w}$, and $\lim_{n\to\infty} f(\underline{u_n}) = f(\lim_{n\to\infty} \underline{u_n}) = f(\underline{w})$ since f is a contraction mapping and therefore Lipschitz continuous. Then $\underline{w} = f(\underline{w})$, so $\underline{w}$ is a fixed point of f.

It remains to establish the uniqueness of $\underline{w}$. Suppose that $\exists \underline{v} \in U$ such that $\underline{v} = f(\underline{v})$. Then $\|\underline{w} - \underline{v}\| = \|f(\underline{w}) - f(\underline{v})\| \leq K\|\underline{w} - \underline{v}\|$, so $(1-K)\|\underline{w} - \underline{v}\| \leq 0$. Now, $(1-K) > 0$ and $\|\underline{w} - \underline{v}\| \geq 0$, so $\|\underline{w} - \underline{v}\| = 0$, so $\underline{w} = \underline{v}$ since $\|\cdot\|$ separates points. ■

Example: Let $\alpha \in (0,1)$. Then, using the Contraction Mapping Theorem, it can be shown that the equation $x = e^{-\alpha x}$, $x \in [0,1]$, has a unique solution $x^* \in (0,1)$.

Now, $(\mathbb{R}, |\cdot|)$ is Banach, and $U = [0,1]$ is closed. It remains to show that $f : [0,1] \to [0,1]$, $f : x \in [0,1] \mapsto e^{-\alpha x}$, is a contraction mapping. $\forall x, y \in [0,1]$, $|f(x) - f(y)| = |f'(\xi)(x-y)| = |-\alpha e^{-\alpha\xi}(x-y)| \leq \alpha|x-y|$ for some $\xi \in (x,y)$ by the Mean Value Theorem, which can be applied since f is continuously differentiable. Then, since $\alpha \in (0,1)$, f is a contraction mapping, so the Contraction Mapping Theorem can be applied, so $\exists! x^* \in (0,1)$ such that $x^* = f(x^*)$. Then x^* solves the equation $x = e^{-\alpha x}$.

However, the Contraction Mapping Theorem not only guarantees the existence of x^*, it provides a method for constructing it, or at least an approximation to it. Let $\alpha = \frac{1}{2}$ and take $x_0 = \frac{1}{2}$. Then

$$\begin{aligned}
x_0 &= \frac{1}{2} \\
x_1 &= e^{-\frac{1}{4}} \approx 0.78 \\
x_2 &\approx 0.68 \\
x_3 &\approx 0.71 \\
x_4 &\approx 0.70 \\
x_5 &\approx 0.70
\end{aligned}$$

It happens that $x_\infty = x^* = W(\frac{1}{2})$, where W is Lambert's W-function, with $W(\frac{1}{2}) \approx 0.70$.

12.2 The Existence and Uniqueness of Solutions to ODEs

Definition: An *ordinary differential equation (ODE)* is of the form

$$\left[\frac{dy}{dt}(t) =\right] \dot{y}(t) = F(t, y(t)),\ t \in A \subset \mathbb{R}$$
$$y(t_0) = y_0,\ t_0 \in A$$

with $y(t)$ the unknown to be solved for subject to the conditions stated. The second equality is known as the *intial condition(s)*.

Remark: Applying $\int_{t_0}^{t}$ to the first equality in the ODE yields

$$\int_{t_0}^{t} \dot{y} = y(t) - y(t_0) = \int_{t_0}^{t} F(\tau, y(\tau))\ d\tau$$

so

$$y(t) = y(t_0) + \int_{t_0}^{t} F(\tau, y(\tau))\ d\tau$$

Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and that $y \in C[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$. Then $y(t_0) + \int_{t_0}^{t} F(\tau, y(\tau))\ d\tau$ is well-defined and differentiable, and so $y(t)$ solves the ODE.

Definition: The operator

$$T(f) := y_0 + \int_{t_0}^{t} F(\tau, f(\tau))\ d\tau$$

acting from $C[t_0 - \delta, t_0 + \delta]$ to $C[t_0 - \delta, t_0 + \delta]$ is called the *Pickard operator*. An *operator* is simply a mapping from one vector space to another, and in many contexts they are linear, however the Pickard operator is not.

Remark: A fixed point of T occurs when $f = T(f)$, so

$$f(t) = y_0 + \int_{t_0}^{t} F(\tau, f(\tau))\ d\tau$$

so f solves the ODE.

Theorem 47 - The Picard-Lindelöf Theorem: Suppose that $F \in C[t_0 - a, t_0 + a] \times [y_0 - \varepsilon, y_0 + \varepsilon]$ for some $a, \varepsilon > 0$, and that F is Lipschitz continuous on $D = [t_0 - a, t_0 + a] \times [y_0 - \varepsilon, y_0 + \varepsilon]$ with respect to y, that is, $\exists L > 0$ such that, $\forall (t, y), (t, y') \in D, \left|F(t, y) - F(t, y')\right| \leq L\left|y - y'\right|$. Then $\exists \delta \in (0, a]$ such that the ODE as given in the definition has a unique solution in $C[t_0 - \delta, t_0 + \delta]$.

Remark: Suppose that

$$\frac{\partial F}{\partial y} \in C(D)$$

Then, $\forall (t, y)(t, y') \in D$ and for some $\xi(t, y, y') \in D$,

$$\left|F(t, y) - F(t, y')\right| \overset{\text{MVT}}{=} \left|\frac{\partial F}{\partial y}(\xi(t, y, y'))\left|y - y'\right|\right| \leq \left\|\frac{\partial F}{\partial y}\right\|_\infty \left|y - y'\right|$$

so F is Lipschitz continuous with respect to y, with

$$L = \left\|\frac{\partial F}{\partial y}\right\|_\infty$$

Proof: The aim of the proof is to find some $\delta > 0$ such that T acts on $C[t_0 - \delta, t_0 + \delta]$ and is a contraction mapping for some closed $U \subset C[t_0 - \delta, t_0 + \delta]$.

For any $\delta > 0$, $T : C[t_0 - \delta, t_0 + \delta] \to C[t_0 - \delta, t_0 + \delta]$. Let y_0 represent the constant function of value y_0, and consider $\overline{B}(y_0, \varepsilon) = \{f \in C[t_0 - \delta, t_0 + \delta] : \|f - y_0\|_\infty \le \varepsilon\} \subset C[t_0 - \delta, t_0 + \delta]$. For any $f \in \overline{B}(y_0, \varepsilon)$,

$$\left\|T(f) - y_0\right\|_\infty = \left\|y_0 + \int_{t_0}^{t} F(\tau, f(\tau))\, d\tau - y_0\right\|_\infty \le \sup_{(\tau,y)\in D} |F| \cdot |t - t_0| \le \sup_{(\tau,y)\in D} |F| \cdot \delta$$

So choose $\delta > 0$ sufficiently small such that

$$\sup_{(\tau,y)\in D} |F| \cdot \delta \le \varepsilon$$

This can be done since

$$\sup_{(\tau,y)\in D} |F| < \infty$$

because F is continuous on D and D is bounded. Then $T : \overline{B}(y_0, \varepsilon) \to \overline{B}(y_0, \varepsilon)$.

Now, $\forall f, g \in \overline{B}(y_0, \varepsilon)$,

$$\begin{aligned}
\left\|T(f) - T(g)\right\|_\infty &= \left\|\int_{t_0}^{t} [F(\tau, f(\tau)) - F(\tau, g(\tau))]\, d\tau\right\|_\infty \\
&\le L\left\|\int_{t_0}^{t} \left|f(\tau) - g(\tau)\right|\, d\tau\right\|_\infty \\
&\le L\|f - g\|_\infty \|t - t_0\|_\infty \\
&\le L\delta\|f - g\|_\infty
\end{aligned}$$

So choose $\delta > 0$ sufficiently small such that $L\delta < 1$ also. Then

1. $(C[t_0 - \delta, t_0 + \delta], \|\bullet\|_\infty)$ is Banach.
2. $\overline{B}(y_0, \varepsilon) \subset C[t_0 - \delta, t_0 + \delta]$ is closed.
3. $T : \overline{B}(y_0, \varepsilon) \to \overline{B}(y_0, \varepsilon)$ is a contraction mapping.

Then, by the Contraction Mapping Theorem, T has a unique fixed point $y \in C[t_0 - \delta, t_0 + \delta]$ such that

$$y(t) = y_0 + \int_{t_0}^{t} F(\tau, y(\tau))\, d\tau$$

which solves the ODE as given in the definition. ■

Note: Because of the Contraction Mapping Theorem, it is possible to construct this solution iteratively. This process is known as *Picard's iteration*. The iteration is given by $y^{(n+1)}(t) = T(y^{(n)}(t))$.

Example: Consider the ODE

$$\begin{aligned}
\dot{y}(t) &= \sqrt{y(t)} \\
y(0) &= 0
\end{aligned}$$

Then $y_1 \equiv 0$ is a solution. However, $y_2(t) = \frac{t^2}{4}$ is also a solution. This can occur because F is not Lipschitz in this case, $\sqrt{y}$ is much greater than y close to 0, so there is no L which bounds the difference between these functions.

Example: Consider the ODE

$$\begin{aligned}\dot{y}(t) &= 2ty(t)\\ y(0) &= 1\end{aligned}$$

with $D = [-1, 1] \times [0, 2]$. Then

$$\begin{aligned}F(t, y) &= 2ty\\ (t_0, y_0) &= (0, 1)\end{aligned}$$

This satisfies the conditions of the Picard-Lindelöf Theorem, so a unique solution exists, and can be constructed iteratively using Picard's iteration, in this case

$$y^{(n+1)} = T(y^{(n)}) = y_0 + \int_0^t 2\tau y^{(n)}(\tau)\, d\tau$$

Now, taking $y^{(0)}(t) = y_0 = 1$ the initial approximation to the solution,

$$\begin{aligned}y^{(0)} &= 1\\ y^{(1)} &= 1 + \int_0^t 2\tau y^{(0)}(\tau)\, d\tau = 1 + \int_0^t 2\tau\, d\tau = 1 + t^2\\ y^{(2)} &= 1 + \int_0^t 2\tau y^{(1)}(\tau)\, d\tau = 1 + \int_0^t 2\tau(1+\tau^2)\, d\tau = 1 + \int_0^t 2\tau + 2\tau^3\, d\tau = 1 + t^2 + \frac{t^4}{2}\\ &\vdots\end{aligned}$$

Using an inductive argument, it can be shown that

$$y^{(n)}(t) = 1 + t^2 + \frac{t^4}{2} + \ldots + \frac{t^{2n}}{n!} \quad \left[= \sum_{k=0}^{n} \frac{t^{2k}}{k!}\right]$$

since $y^{(0)}$ is of this form, and, assuming the statement holds for $y^{(n)}$,

$$\begin{aligned}y^{(n+1)}(t) &= 1 + \int_0^t 2\tau y^{(n)}(\tau)\, d\tau\\ &= 1 + \int_0^t 2\tau \left(1 + \tau^2 + \frac{\tau^4}{2} + \ldots + \frac{\tau^{2n}}{n!}\right) d\tau\\ &= 1 + \int_0^t 2\tau + 2\tau^3 + \tau^5 + \ldots + \frac{2\tau^{2n}}{n!}\, d\tau\\ &= 1 + t^2 + \frac{t^4}{2} + \ldots + \frac{t^{2n}}{n!} + \frac{t^{2(n+1)}}{(n+1)!}\end{aligned}$$

which is of the correct form. Then, with respect to $\|\cdot\|_\infty$,

$$y(t) = \lim_{n\to\infty} y^{(n)}(t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} = e^{t^2}$$

which satisfies the ODE.

Exercise: Check the existence and uniqueness of a solution to this ODE on $C[-\delta, \delta]$, where $\delta \in (0, \frac{1}{4})$.

12.3 The Jacobi Algorithm

Motivation: Let $A \in \mathbb{R}^{n\times n}$ be an $n \times n$ real-valued matrix, and let $\underline{b} \in \mathbb{R}^n$ be a real-valued n-dimensional vector. Then, if $\det A \neq 0$, the equation $A\underline{x} = \underline{b}$ has the unique solution $\underline{x} = A^{-1}\underline{b}$ for some $\underline{x} \in \mathbb{R}^n$. However, inverting A is computationally expensive, typically $O(n^3)$. Whilst not a problem for small n, if, for example, $n = 10^8$, this quickly becomes impractical to compute.

However, if $A = D + R$ for some $D, R \in \mathbb{R}^{n\times n}$, where D is diagonal and non-degenerate ($\det D \neq 0$) and R is "small", so

$$A = \begin{pmatrix} d_{11} & 0 & \dots & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & d_{nn} \end{pmatrix} + \begin{pmatrix} 0 & r_{12} & \dots & \dots & r_{1n} \\ r_{21} & 0 & r_{23} & \dots & r_{2n} \\ \vdots & r_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & r_{n-1n} \\ r_{n1} & r_{n2} & \dots & r_{nn-1} & 0 \end{pmatrix}$$

then $D\underline{x} + R\underline{x} = \underline{b}$, so $D\underline{x} = \underline{b} - R\underline{x}$. If R is "small", then $D\underline{x} \approx \underline{b}$. Let $\underline{x^{(n)}} \subset \mathbb{R}^n$ be a sequence of approximations to $\underline{x}$. Then it seems reasonable to set $\underline{x^{(0)}} = D^{-1}\underline{b}$, and let

$$\begin{aligned} \underline{x^{(1)}} &= D^{-1}(\underline{b} - R\underline{x^{(0)}}) \\ \underline{x^{(2)}} &= D^{-1}(\underline{b} - R\underline{x^{(1)}}) \\ &\vdots \\ \underline{x^{(n+1)}} &= D^{-1}(\underline{b} - R\underline{x^{(n)}}) \end{aligned}$$

This is called the *Jacobi iteration.* Using the Contraction Mapping Theorem, under certain conditions it can be shown that this sequence converges to $\underline{x}$. Since diagonal inversion is only of order $O(n)$, this is very fast to compute, so these approximations are extremely useful in situations where inverting A is not practical. Additionally, the sequence converges to $\underline{x}$ exponentially.

Proposition: Suppose that

$$\left\|D^{-1}R\right\| := \sup_{\underline{w}\neq 0} \frac{\left\|D^{-1}R\underline{w}\right\|_2}{\|\underline{w}\|_2} = \lambda < 1$$

Then the Jacobi iteration as defined above converges to $\underline{x}$, the solution to $A\underline{x} = \underline{b}$.

Proof: Let $T : \mathbb{R}^n \to \mathbb{R}^n$, $T : \underline{x} \mapsto D^{-1}(\underline{b} - R\underline{x})$. T acts on $(\mathbb{R}^n, \|\cdot\|_2)$, which is a Banach space, and $\mathbb{R}^n \subseteq \mathbb{R}^n$ is closed. Now, $\forall \underline{u}, \underline{v} \in \mathbb{R}^n$,

$$\begin{aligned} \|T\underline{u} - T\underline{v}\|_2 &= \left\|(D^{-1}\underline{b} - D^{-1}R\underline{u}) - (D^{-1}\underline{b} - D^{-1}R\underline{v})\right\|_2 = \left\|D^{-1}R(\underline{u} - \underline{v})\right\|_2 \\ &\leq \left\|D^{-1}R\right\| \|\underline{u} - \underline{v}\|_2 = \lambda \|\underline{u} - \underline{v}\|_2 \end{aligned}$$

Then, since $\lambda < 1$, T is a contraction mapping on $\mathbb{R}^n$, so, by the Contraction Mapping Theorem, T has a unique fixed point $\underline{h} \in \mathbb{R}^n$. $\underline{h} = T(\underline{h}) = D^{-1}(\underline{b} - R\underline{h})$, so $D\underline{h} = \underline{b} - R\underline{h}$, so $\underline{(D + R)(\underline{h})} = \underline{b}$, so $A\underline{h} = \underline{b}$, so the Jacobi iteration constructs a solution to $A\underline{x} = \underline{b}$. Moreover

$$\left\|\underline{h} - \underline{x^{(n)}}\right\|_2 = \left\|T(\underline{h}) - T\left(\underline{x^{(n-1)}}\right)\right\|_2 \leq \lambda \left\|\underline{h} - \underline{x^{(n-1)}}\right\|_2 \leq \dots \leq \lambda^n \left\|\underline{h} - \underline{x^{(0)}}\right\|_2 = e^{-n \log \frac{1}{\lambda}} \left\|\underline{h} - D^{-1}\underline{b}\right\|_2$$

so the sequence converges exponentially, and the number of iterations needed does not depend on the dimension of $\mathbb{R}^n$. ■

12.4 The Newton-Raphson Algorithm

Motivation: For some continuously differentiable $f : \mathbb{R} \to \mathbb{R}$, finding an $x \in \mathbb{R}$ satisfying the equation $f(x) = 0$ is non-trivial, and a closed-form expression for x cannot always be found, quintic polynomials are a simple example of this. It would be useful then to construct a series approximating x, and converging to this solution.

Initially, choosing some $x^{(0)}$ "close" to the root, then taking the Taylor expansion about this point yields $g(x) \approx g(x^{(0)}) + g'(x^{(0)})(x - x^{(0)})$. Now, for some $x^{(1)} \in \mathbb{R}$, $0 \approx g(x^{(1)}) = g(x^{(0)}) + g'(x^{(0)})(x^{(1)} - x^{(0)})$, so it seems reasonable to set

$$x^{(1)} - x^{(0)} = -\frac{g(x^{(0)})}{g'(x^{(0)})}$$

then

$$x^{(1)} = x^{(0)} - \frac{g(x^{(0)})}{g'(x^{(0)})}$$

Now, let

$$x^{(n+1)} = N(x^{(n)}) := x^{(n)} - \frac{g(x^{(n)})}{g'(x^{(n)})}$$

This is called the *Newton-Raphson method.* $N : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$, and is called the *Newton-Raphson operator.* $(x^{(n)})_{n \geq 0} \subset \mathbb{R}$ is a sequence of approximations to x, and, using the Contraction Mapping Theorem, under certain conditions it can be shown that this sequence converges to x.

Note: This sequence does not always converge, especially if the initial guess is poor. For example, if f is concave on the right and convex on the left, then the sequence can diverge if the initial guess is particularly bad.

Proposition: For any continuously differentiable $f : \mathbb{R} \to \mathbb{R}$, if

$$\left\| \frac{f \cdot f''}{(f')^2} \right\|_\infty = \lambda < 1$$

then the Newton-Raphson method as defined above converges to x, the solution to $f(x) = 0$.

Proof: $\forall x, y \in \mathbb{R}$ and for some $\xi \in (x, y)$,

$$\begin{aligned} \big|N(x) - N(y)\big| &\overset{\text{MVT}}{=} \big|N'(\xi)|x-y|\big| = \left| \left(1 - \frac{f'(\xi)}{f'(\xi)} + \frac{f(\xi) \cdot f''(\xi)}{f'(\xi)^2}\right)|x-y| \right| = \left| \frac{f(\xi) \cdot f''(\xi)}{f'(\xi)^2} \right| |x-y| \\ &\leq \left\| \frac{f \cdot f''}{(f')^2} \right\|_\infty |x-y| = \lambda |x-y| \end{aligned}$$

Then, since $\lambda < 1$, N is a contraction mapping on $\mathbb{R}$, so, by the Contraction Mapping Theorem, N has a unique fixed point $h \in \mathbb{R}$.

$$h = N(h) = h - \frac{f(h)}{f'(h)}$$

so

$$h - h = -\frac{f(h)}{f'(h)}$$

so $f(h) = 0$. ■