A Fubini Counterexample

We attempt to evaluate the double integral

$$\iint_R f(x,y)dx\,dy$$

over the rectangle R given by $0 \le x \le 2$, $0 \le y \le 1$, with the function f defined by

$$f(x,y) = \frac{xy(x^2-y^2)}{(x^2+y^2)^3} \quad \text{for } (x,y) \ne (0,0)$$

and $f(0,0) = 0$.

First approach We integrate first in the y-direction. For any x, we define $A(x) = \int_0^1 f(x,y)dy$. For now, we assume that $x \ne 0$. We make the substitution $u = x^2 + y^2$, $du = 2y\,dy$, use $x^2 - y^2 = x^2 - (u - x^2) = 2x^2 - u$, and compute

$$A(x) = \int_{x^2}^{x^2+1} \frac{x(2x^2-u)du}{2u^3} = \int_{x^2}^{x^2+1} \left(\frac{x^3}{u^3} - \frac{x}{2u^2}\right) du = \left[-\frac{x^3}{2u^2} + \frac{x}{2u}\right]_{u=x^2}^{u=x^2+1}$$

$$= -\frac{x^3}{2(x^2+1)^2} + \frac{x}{2(x^2+1)} + \frac{1}{2x} - \frac{1}{2x} = \frac{x}{2(x^2+1)^2}$$

We note that this formula remains valid for $x = 0$, as f vanishes on the whole y-axis.

Then we integrate in the x-direction,

$$\int_0^2 A(x)dx = \int_0^2 \frac{x\,dx}{2(x^2+1)^2} = \left[\frac{-1}{4(x^2+1)}\right]_0^2 = -\frac{1}{20} + \frac{1}{4} = \frac{1}{5}$$

Second approach We integrate first in the x-direction. For any y, we define $B(y) = \int_0^2 f(x,y)dx$. If $y \ne 0$, we again make the substitution $u = x^2 + y^2$, except now $du = 2x\,dx$. In view of the (skew-)symmetry of f, the integral is almost the same,

$$B(y) = \left[-\frac{y}{2u} + \frac{y^3}{2u^2}\right]_{u=y^2}^{u=4+y^2} = -\frac{y}{2(4+y^2)} + \frac{y^3}{2(4+y^2)^2} = \frac{-2y}{(4+y^2)^2}$$

This formula too remains valid for $y = 0$, as f vanishes on the x-axis.

Then we integrate in the y-direction,

$$\int_0^1 B(y)dy = \int_0^1 \frac{-2y\,dy}{(4+y^2)^2} = \left[\frac{1}{4+y^2}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

which disagrees with our first answer. Worse, this had to happen because $B(y)$ is negative, while $A(x)$ is positive.

Post-mortem What happened here? The fact that all the functions we integrated are continuous functions of one variable offers no clue that anything is wrong. The point is that Fubini's Theorem *does not apply*, because the function f is not integrable over R; indeed, it is not even bounded on R. (Nor is it Lebesgue-integrable.) It is continuous away from 0 but has a bad discontinuity at 0. What makes this counterexample work is that f takes arbitrarily large positive and negative values near the origin; explicitly, $f(2t,t) = 6/125t^2$ and $f(t,2t) = -6/125t^2$ for any $t \ne 0$.