Thm 24' $\frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}\sin mx\,dx}_{0} + \sum_{k=1}^{\infty} a_k\Big(\underbrace{\int_{-\pi}^{\pi}\sin mx\cos kx\,dx}_{\frac{1}{4i}\int_{-\pi}^{\pi}dx\,(e^{imx}-e^{-imx})(e^{ikx}+e^{-ikx})=0}$

$$+\ b_k\underbrace{\int_{-\pi}^{\pi}\sin mx\sin kx\,dx}_{\left(-\frac{1}{4}\right)\int_{-\pi}^{\pi}(e^{imx}-e^{-imx})(e^{ikx}-e^{-ikx})=\frac{1}{4}2\cdot 2\pi\,\delta_{km}=\pi\delta_{km}}\Big) = \pi b_m$$

Exercise. Show that $\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \pi a_k$ $(k\geq 0)$ #

Remarks ① Recall that Kronecker δ-symbol is

$$\delta_{km}=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}\quad (k,m\in\mathbb{Z}),$$

so that $\sum_{n\in\mathbb{Z}} f_n\delta_{kn} = f_k$

② Given f, Fourier series defined using (*) is said to be generated by f. At the moment, we do not ③

know whether it converges to f, or converges at all. ④

(But:) The great Corollary of Thm. 31 (Fourier representation of functions.) Let $f:[-\pi,\pi]\to\mathbb{R}$ be a C^2-function: $f(-\pi)=f(\pi)$. Then the Fourier series

$$(**)\qquad \frac{a_0}{2}+\sum_{k=1}^{\infty}\big(a_k\cos(kx)+b_k\sin(kx)\big)$$

with $(a_k)_{k\geq 1}$, $(b_k)_{k\geq 1}$ defined by (*) converges to f uniformly on $[-\pi,\pi]$.

Part I: (**) converges
Part II: the limit is f

Proof. (Part I) $\forall k\geq 1$: $\int_{-\pi}^{\pi} f(x)\cos kx\,dx$

$$=\frac{1}{k}\int_{-\pi}^{\pi} f(x)(\sin kx)'\,dx$$

$$=\frac{1}{k}\Big(\underbrace{f(x)\sin(kx)\Big|_{-\pi}^{\pi}}_{0}-\int_{-\pi}^{\pi} f'(x)\sin(kx)\,dx$$

f', f'' - continuous on $[-\pi,\pi]$

$$=-\frac{1}{k}\int_{-\pi}^{\pi} f'(x)\cos'(kx)\,dx$$

Heat equation $\frac{\partial T}{\partial t}(\varphi,t) - \frac{\partial^2 T}{\partial \varphi^2}(\varphi,t) = 0 \quad t>0, \quad \varphi \in (-\pi,\pi)$

Periodic boundary conditions
$$\begin{cases} T(-\pi,t) = T(\pi,t), & t>0 \\ \frac{\partial T}{\partial \varphi}(-\pi,t) = \frac{\partial T}{\partial \varphi}(\pi,t) \end{cases}$$

Initial condition $T(\varphi, 0) = f(\varphi) \in C^2(S^1)$

Notice: (i) $\{1, (\cos k\varphi, \sin k\varphi)_{k\geq 1}\}$ satisfy periodic boundary conditions

(ii) $\frac{\partial^2}{\partial \varphi^2}\begin{Bmatrix}\cos k\varphi \\ \sin k\varphi\end{Bmatrix} = -k^2 \begin{Bmatrix}\cos k\varphi \\ \sin k\varphi\end{Bmatrix}$

(iii) $f(\varphi)$ can be decomposed into Fourier series.

Fourier's idea: look for solution in the form of Fourier series.

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$$(*) \quad T(\varphi,t) = \frac{a_0(t)}{2} + \sum_{k=1}^{\infty}\left(a_k(t)\cos k\varphi + b_k(t)\sin k\varphi\right)$$

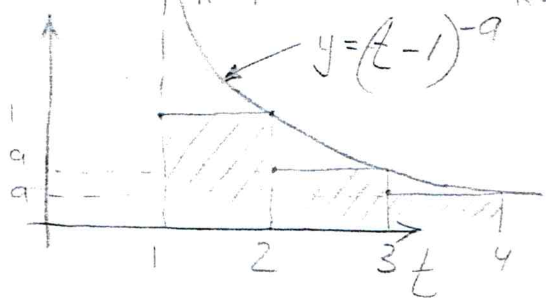
Assume that Fourier series representing $\dot{T}$, T' and T'' converge uniformly. Then we can differentiate $T(\varphi,t)$ termwise. Substituting $(*)$ into the heat equation we get

$\dot{T} \equiv \frac{\partial T}{\partial t}$

$T' \equiv \frac{\partial T}{\partial \varphi}$

$$\frac{\dot{a}_0(t)}{2} + \sum_{k=1}^{\infty}\Big(\left(\dot{a}_k(t) + k^2 a_k(t)\right)\cos k\varphi + \left(\dot{b}_k(t) + k^2 b_k(t)\right)\sin k\varphi\Big) = 0$$

$$\sum_{k=1}^{n} \frac{1}{k^x} \le \sum_{k=1}^{n} \frac{1}{k^a} \le 1 + \int_2^n \frac{1}{(t-1)^a}\,dt$$



$$= 1 + \int_1^{n-1} \frac{1}{t^a}\,dt = 1 + \frac{t^{1-a}}{1-a}\Big|_1^{n-1}$$

$$\longrightarrow 1 - \frac{1}{1-a} = \frac{a}{a-1} < \infty$$

$\underset{\text{test}}{\overset{M}{\Rightarrow}}$ $\sum_{k=1}^{\infty} \frac{1}{k^x}$ converges uniformly on $[a,\infty)$

As $f_k(x) = \frac{1}{k^x}$ is continuous on $[a,\infty)$,

$\zeta(x) = \sum_{k=1}^{\infty} k^{-x}$ is continuous on $[a,\infty)$.

We established that $\forall a>1$, ζ exists and is continuous at every point of $[a,\infty)$ $\Rightarrow$ ζ exists and is continuous on $(1,\infty)$

Remarks. (1) $\sum_k k^{-x}$ does not converge uniformly on $(1,\infty)$ (Exercize)

(2) $\frac{d}{dx} k^{-x} = \begin{cases} 0 & k=1 \\ -\log k\, k^{-x} & k>1 \end{cases}$ and $\forall a>1$ $\sum_k \log k\, k^{-a} < \infty$

$\underset{M\text{-test}}{\overset{\text{Thm 24}}{\Longrightarrow}}$ ζ is C^1 on $(1,\infty)$ (In fact C^∞: exercize)

End of p. 9