

15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to two randomly chosen 'A' and two 'B' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' questions.

The answers to ALL questions are to be submitted by the deadline of 3pm on Monday 20st October 2014. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeonloft if you are a Maths student, or in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

## 0.1 Step functions. Integration of step functions.

- A. Function  $\psi : [a, b] \rightarrow \mathbb{R}$  is called a *squelch function* if for **each** partition  $a = p_0 < p_1 < \dots < p_{k-1} < p_k = b$ ,  $\psi$  is constant on each open interval  $(p_{j-1}, p_j)$ ,  $1 \leq j \leq k$ .

(i) What is the maximal possible cardinality of the set of values of  $\psi$ ? (ii) Is  $\psi$  a step function? (iii) True or false: any step function on  $[a, b]$  is also a squelch function. Justify your answers.
- B. (i) Prove that an arbitrary linear combination of step functions  $\phi_1, \phi_2, \dots, \phi_n \in S[a, b]$  is a step function. (ii) Let  $\{\phi_k\}_{k=1}^{\infty}$  be an infinite sequence of step functions on  $[a, b]$ . Suppose that the series  $\sum_{k=1}^{\infty} \phi_k(x)$  converges at every  $x \in [a, b]$ . Define  $\phi(x) := \sum_{k=1}^{\infty} \phi_k(x)$ . True or false:  $\phi \in S[a, b]$ .
- A. Find an example of a function  $\phi : [0, 1] \rightarrow \mathbb{R}$  such that:  $\phi$  is not a step function, but  $|\phi|$  is a step function.
- A. Suppose  $h$  is a step function on  $[a, b]$  and that  $P = \{z_0 < z_1 < \dots < z_n\}$  is a partition of  $[a, b]$  for which  $h$  is constant on each subinterval  $(z_{i-1}, z_i)$ . (i) Prove that

$$\int_a^b h = \sum_{i=1}^n h(w_i)(z_i - z_{i-1}),$$

where, for each  $1 \leq i \leq n$ ,  $w_i$  is any point in  $(z_{i-1}, z_i)$ . (If you are familiar with the Riemann theory of integration, you will notice that the integral of a step function takes the form of a Riemann sum.) (ii) Show that  $\int h$  is independent of the values of  $h$  at the points  $\{z_i\}$  of the partition  $P$ .

- B. Let  $h_1$  and  $h_2$  be two step functions on  $[a, b]$ . (i) Suppose that  $h_1(x) = h_2(x)$  for all  $x \in [a, b]$  except for one point  $c$ . Prove that

$$\int_a^b h_1 = \int_a^b h_2.$$

(ii) Suppose  $h_1(x) = h_2(x)$  for all but a finite number of points  $c_1, \dots, c_N \in [a, b]$ . Prove that

$$\int_a^b h_1 = \int_a^b h_2.$$

- A. Let  $h_1, h_2 : [a, b] \rightarrow \mathbb{R}$  be step functions. Prove that if  $h_1 \geq h_2$ , then  $\int_a^b h_1 \geq \int_a^b h_2$ .
- A. Let  $\varphi, \psi : [a, b] \rightarrow \mathbb{R}$  be step functions.

(i) Prove that  $|\varphi|$  is a step function and that  $|\int_a^b \varphi| \leq \int_a^b |\varphi|$ .

(ii) Prove that  $\int_a^b |\varphi + \psi| \leq \int_a^b |\varphi| + \int_a^b |\psi|$ .
- A. Define the step function  $\varphi : [-5, 5] \rightarrow \mathbb{R}$  by  $\varphi(x) = -1$ , for  $-5 \leq x \leq 0$ , and  $\varphi(x) = +1$ , for  $0 < x \leq 5$ . What is  $\Phi : [-5, 5] \rightarrow \mathbb{R}$ ,  $\Phi(x) := \int_{-5}^x \varphi$ ? Show that  $\Phi$  is differentiable, except at 0, with derivative  $\varphi$ .
- B. (Poor man's fundamental theorem of calculus.) Let  $\phi \in S[a, b]$ . Let  $P$  be a partition of  $[a, b]$ ,  $a = p_0 < p_1 < p_2 < \dots < p_N = b$ , such that  $\phi|_{(p_{i-1}, p_i)} = \phi_i = \text{const}$ ,  $i = 1, 2, \dots, N$ . Denote  $\phi(a) = \phi_0$ . Assume that  $\phi$  is continuous at  $x = b$ , i. e.  $\phi(b) = \phi_N$ . Let us define the function  $D\phi$  on  $[a, b]$ :

$$D\phi|_{(p_{i-1}, p_i)} = \frac{\phi_i - \phi_{i-1}}{p_i - p_{i-1}}, \quad i = 1, 2, \dots, N.$$

Assign any real values you like to  $D\phi(p_i)$ . Clearly,  $D\phi$  is a step function. (Notice however that its construction does depend on  $P$ .) Calculate  $\int_a^b D\phi(x) dx$ .

10. B. Let  $\varphi, \psi : [a, b] \rightarrow \mathbb{R}$  be step functions.  
 (a) Prove that their product  $\varphi\psi : [a, b] \rightarrow \mathbb{R}$  is a step function.  
 (b) Prove that  $(\int_a^b \varphi\psi)^2 \leq (\int_a^b \varphi^2)(\int_a^b \psi^2)$ .  
 [This is called the Cauchy-Schwarz inequality for the integral of step functions.  
 Suggestion: consider the quadratic function of  $t$  defined by  $\int_a^b (t\varphi + \psi)^2$ .]

## 0.2 Regulated functions. Integration of regulated functions

11. A. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be given by  $f(0) = 0$ ,  $f(x) = x \cdot \sin(1/x)$  for  $x > 0$ . Prove that  $f$  is regulated.
12. A. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be given by  $f(0) = 0$ ,  $f(x) = 1/x$  for  $x > 0$ . Prove that  $f$  is not regulated. .
13. B. Let  $f : [a, b] \rightarrow \mathbb{R}$  be regulated and non-negative. Prove that  $g : [a, b] \rightarrow \mathbb{R}$  defined by  $g(x) = \sqrt{f(x)}$  is regulated.
14. A. Let  $a > 0$ . By using a suitable sequence of step functions on  $[0, a]$ , show directly from the definition that  $\int_0^a dx x = a^2/2$ .
15. A. By using a suitable sequence of step functions  $\phi_n$  for the partition  $P = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ , show directly from the definition that  $\int_0^1 e^x dx = e - 1$ .
16. B. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be regulated. Let  $g : [0, 1] \rightarrow \mathbb{R}$  be equal to  $f$  at every point of  $[0, 1]$  except for  $x_1, x_2, \dots, x_k \in [0, 1]$  where it is equal to  $g_1, g_2, \dots, g_k$  correspondingly. Prove that  $g$  is regulated and that  $\int_0^1 f = \int_0^1 g$ .
17. A. If the regulated function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  is even (meaning that  $\forall x \in [-\pi, \pi] f(x) = f(-x)$ ) show that  $\int_{-\pi}^{\pi} f = 2 \int_0^{\pi} f$ .
18. B. (Riemann-Lebesgue lemma.) For  $p < q$  in  $\mathbb{R}$  show that (i)  $\int_p^q \sin(tx) dx \rightarrow 0$  as  $t \rightarrow \infty$ . (ii) For  $\varphi \in S[a, b]$  show that  $\int_a^b \varphi(x) \cdot \sin(tx) dx \rightarrow 0$  as  $t \rightarrow \infty$ . [Suggestion: treat separately each of the intervals on which  $\varphi$  is constant.] (iii) For  $f \in R[a, b]$  show that  $\int_a^b f(x) \cdot \sin(tx) dx \rightarrow 0$  as  $t \rightarrow \infty$ .
19. B. Let  $\mathbb{R}[x]$  denote the real vector space of polynomials in  $x$  with real coefficients, and let  $D, I : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$  be given by  $D(p) := p'$  and  $I(p) := q$  where  $q(x) := \int_0^x p$  (using the integration you have studied before). Show that  $D, I$  are linear maps. Are they surjective or injective; what is the kernel? Is it true that  $D \circ I = I \circ D = id_{\mathbb{R}[x]}$ ?
20. A. Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Show that there exists  $c \in (a, b)$  so that

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c).$$

[Hint: one method is to use the Mean Value Theorem.]