Questions for credit: 5 (6 points), 14 (6 points), 15 (6 points) and 18 (7 points)

0.1 Step functions. Integration of step functions.

1. (i) Let $P = \{a, b\}$, $k = 1$ and $a = p_0 < p_1 = b$. By definition of squelch functions, $\psi|_{(a,b)} = c$, where $c \in \mathbb{R}$ is a constant. Therefore, $Image(\psi) = \{\psi(a), \psi(b), c\}$. The maximal cardinality of this set is 3. (ii) P is a partition of $[a, b]$ such that ψ is constant on the only open subinterval this partition. So ψ is a step function. (iii) No, $f(x) = sign(x)$, $x \in [-1, 1]$ is a step function on $[-1, 1]$ but not a squelch function.

2. (i) The result is proved by induction using the fact that for any $\phi, \psi \in S[a, b]$ and $\lambda, \mu \in \mathbb{R}$, $\lambda\phi + \mu\psi \in S[a, b]$. This fact was proved in the lecture: if P is a partition of $[a, b]$ compatible with ϕ and Q is a partition of $[a, b]$ compatible with ψ, then the refined partition $P \cup Q$ is compatible with $\lambda\phi + \mu\psi$. (ii) False: let $\phi_n(x) = 1/n$ for $x \in (1 - 1/n, 1 - 1/(n+1))$, $n = 1, 2, 3, \ldots$ and zero otherwise. Clearly, $\phi_n \in S[0, 1]$. But $\phi := \sum_{k=1}^{\infty} \phi_n$ is not a step function as for any partition $0 = p_0 < p_1 < \ldots < p_{k-1} < p_k = 1$, $\phi \mid_{(p_{k-1}, p_k)}$ is not constant.

3. Define $\phi(x) = 1$ if x is rational, $\phi(x) = -1$ if x is irrational. Clearly, ϕ is not a step function, as it is not constant on any non-empty open subinterval of $[0, 1]$. (Students should still present a full argument.) But $|\phi| \equiv 1$ - a constant, hence a step function.

4. (i) If $h \mid_{(z_{i-1}, z_i)} = c_i$ - constant, then $h(w_i) = c_i$ for any $w_i \in (z_{i-1}, z_i)$. By definition of $\int h$,
$$\int_a^b h = \sum_{i=1}^{n} c_i(z_i - z_{i-1}) = \sum_{i=1}^{n} h(w_i)(z_i - z_{i-1}).$$
(ii) Changing the value of h at $z_1, z_2, \ldots z_n$ does not change $h(w_i)$, where $w_i \in (z_{i-1}, z_i)$. Therefore, $\int h$ does not change by part (i).

5. Let P, Q be partitions of $[a, b]$ compatible with step functions h_1 and h_2 correspondingly. Then the refinement $R = P \cup Q \cup \{c\}$ is compatible with both h_1 and h_2. Also, $c \in R$. Let $R = \{z_0, z_1, \ldots z_k\}$. By construction, $h_1(w) = h_2(w)$ for any $w \in (z_{i-1}, z_i)$, $1 \leq i \leq k$. By exercise 4 this implies that $\int_a^b h_1 = \int_a^b h_2$. (ii) The proof is identical to part (i), if one considers the partition $R = P \cup Q \cup \{c_1, c_2, \ldots, c_N\}$

6. Let $a = z_0 < z_1 < z_2 < \ldots < z_k = b$ be a partition of $[a, b]$ compatible with both h_1 and h_2. By exercise 4,
$$\int_a^b h_1 = \sum_{i=1}^{k} h_1(w_i)(z_i - z_{i-1}),$$
$$\int_a^b h_2 = \sum_{i=1}^{k} h_2(w_i)(z_i - z_{i-1}),$$
where $w_i \in (z_{i-1}, z_i)$. As $h_1(w_i) \geq h_2(w_i)$ for any i between 1 and k, we are done.

7. (i) If $\phi \in S[a,b]$, then $|\phi| \in S[a,b]$ (use any partition compatible with ϕ). Therefore, all integrals we are asked to analyze are defined. As $|\phi| \geq \pm\phi$, the previous exercise and the linearity of the integral implies that $\int_a^b |\phi| \geq \pm \int_a^b \phi$, which implies of course that $\int_a^b |\phi| \geq |\int_a^b \phi|$. (ii) As $|\phi+\psi| \leq |\phi|+|\psi|$ (triangle inequality) and $|\phi+\psi|, |\phi|, |\psi| \in S[a,b]$, we have by exercise 6: $\int_a^b |\phi+\psi| \leq \int_a^b(|\phi|+|\psi|) = \int_a^b |\phi| + \int_a^b |\psi|$, where the last equality is due to the linearity of the integral.

8. Note that $\phi\mid_{[-5,x]}$ is a step function for any $x \in [-5,5]$. To calculate integral we can use exercise 4 and the following partitions: $\{-5,x\}$ for $x \leq 0$ and $\{-5,0,x\}$ for $x > 0$. The answer is $\Phi(x) = -5+|x|$. Therefore (Analysis II), Φ is differentiable on $[-5,0)\cup(0,5]$ and $\Phi'(x) = -1 = \phi(x)$ for $x < 0$, $\Phi'(x) = 1 = \phi(x)$ for $x > 0$.

9. By definition of the integral of the step function,

$$\int_a^b D\phi(x)dx = \sum_{k=1}^{N} D\phi\mid_{(p_{i-1},p_i)}(p_i - p_{i-1}) = \sum_{k=1}^{N}(\phi_i - \phi_{i-1}) = \phi_N - \phi_0 = \phi(b) - \phi(a).$$

10. The refinement of partitions compatible with φ and ψ is a partition compatible with the product function $\varphi\psi$. This shows that $\varphi\psi$ is a step function. Then $t\varphi+\psi$ and $(t\varphi+\psi)^2$ are also step functions. So, $\forall t \in \mathbb{R}$,

$$\begin{aligned} 0 &\leq \int_a^b (t\varphi+\psi)^2 = \sum_1^k (tc_j + c_j')^2(p_j - p_{j-1}) \\ &= t^2\sum_1^k c_j^2(p_j - p_{j-1}) + 2t\sum_1^k c_jc_j'(p_j - p_{j-1}) + \sum_1^k c_j'^2(p_j - p_{j-1}) \\ &= t^2\int_a^b \varphi^2 + 2t\int_a^b \varphi\psi + \int_a^b \psi^2 = At^2 + 2Bt + C, \end{aligned}$$

say. Therefore the quadratic polynomial $At^2+2Bt+C$ has at most one real root. Thus

$$\left(\int_a^b \varphi\psi\right)^2 = B^2 \leq AC = \left(\int_a^b \varphi^2\right)\left(\int_a^b \psi^2\right),$$

as required.

0.2 Regulated functions. Integration of regulated functions.

11. f is continuous at $x > 0$ (as the product of two continuous functions). Moreover,

$$\lim_{x \downarrow 0} f(x) = 0 = f(0),$$

so f is continuous at 0. Therefore, f is continuous on $[0,1]$, hence it is regulated.

12. Let us consider an arbitrary step function $\phi \in S[0,1]$. Let $P = \{0, p_1, \ldots, p_{k-1}, 1\}$ be a partition of $[0,1]$ compatible with ϕ. Let $c = \phi\mid_{(0,p_1)}$. As f is unbounded on $(0,p_1)$, it is always possible to find $x \in (0,p_1)$ such that $|f(x)-c| > 1$. Therefore, there exists no step function $\phi \in S[0,1]$ such that $||f-\phi||_\infty < 1$. Therefore, f is not regulated.

13. f is regulated. So for any $\epsilon > 0$ there is $\phi \in S[0,a]$ such that for any $x \in [0,a]$

$$|f(x) - \phi| < \epsilon'.$$

As $f \geq 0$ we can always choose $\phi \geq 0$. (Why?) Let t $P = \{0, p_1, \ldots, p_{k-1}, a\}$ be a partition of $[0,a]$ compatible with ϕ. Let $c_i = \phi\mid_(p_{i-1}, p_i)$, $c_i \geq 0$. For $x \in (p_{i-1}, p_i)$,

$$|f(x) - c_i| < \epsilon'$$

If $c_i = 0$, the above inequality implies that

$$\sqrt{f(x)} < \sqrt{\epsilon'}. \tag{1}$$

If $c_i > 0$, it gives

$$|\sqrt{f} - \sqrt{c_i}| < \frac{\epsilon'}{\sqrt{f} + \sqrt{c_i}} < \frac{\epsilon'}{\sqrt{c_i}}. \tag{2}$$

Now, for any $\epsilon > 0$ we can choose $\epsilon' > 0$ such that

$$\max\left(\max_{i:c_i>0} \frac{\epsilon'}{\sqrt{c_i}}, \sqrt{\epsilon'}\right) < \epsilon$$

Then the inequalities (1, 2) show that

$$||\sqrt{f} - \sqrt{\phi}||_\infty < \epsilon.$$

As $\sqrt{\phi}$ is a step function, the proof is complete. (It is worth verifying that ϕ can be chosen in such a way that nothing goes wrong at the points of the partition.)

14. Let $P_n = \{0, a/n, 2a/n, \ldots a(n-1)/n\}$ be the sequence of partitions of $[0,a]$. Define a sequence of step functions ϕ_n on $[0,a]$: $\phi_n\mid_{(a\frac{(k-1)}{n}, a\frac{k}{n})} = a\frac{k}{n}$. Clearly, $\lim_{n\to\infty} ||x - \phi_n||_\infty = 0$. But

$$\int_0^a \phi_n = \frac{a}{n}\sum_{k=1}^{n} a\frac{k}{n} = \frac{a^2}{n^2}\frac{n(n+1)}{2} \overset{n\to\infty}{\to} \frac{a^2}{2}.$$

Therefore, $\int_0^a dx x = \frac{a^2}{2}$ by definition.

15. The set up is exactly the same as for the previous exercise, but $\phi_n\mid_{(\frac{(k-1)}{n}, \frac{k}{n})} = e^{\frac{(k-1)}{n}}$. Then

$$\int_0^1 dx e^x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{\frac{(k-1)}{n}} = \lim_{n\to\infty} \frac{e-1}{n(e^{1/n} - 1)} = e - 1,$$

where we had to sum a geometric series.

16. For any $\epsilon > 0$ there is $\phi \in S[a,b]$: such that for any $x \in [a,b]$,

$$\phi(x) - \epsilon \leq f(x) \leq \phi(x) + \epsilon. \tag{3}$$

Define $\tilde{\phi} \in S[a,b]$:

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & x \neq x_1, x_2, \ldots, x_k, \\ g_k & x = x_k, k = 1, 2, \ldots, x_k. \end{cases}$$

Then, for any $x \in [a,b]$,

$$\tilde{\phi}(x) - \epsilon \leq f(x) \leq \tilde{\phi}(x) + \epsilon. \tag{4}$$

We know (Exercise 5(ii)), that

$$\int_a^b \phi = \int_a^b \tilde{\phi}. \tag{5}$$

Using (3, 4, 5), we can establish the following two bounds on $\int g$:

$$\int_a^b g \leq \int_a^b (\tilde{\phi} + \epsilon) = \int_a^b \phi + \epsilon(b-a) \leq \int_a^b f + 2\epsilon(b-a)$$

and

$$\int_a^b g \geq \int_a^b (\tilde{\phi} - \epsilon) = \int_a^b \phi - \epsilon(b-a) \geq \int_a^b f - 2\epsilon(b-a).$$

Taking the limit $\epsilon \downarrow 0$ we find that $\int_a^b f = \int_a^b g$.

17. f is regulated. Therefore, the restriction $f\mid_{[0,\pi]}$ is regulated. For any $\epsilon > 0$, there is $\phi \in S[0,\pi]$ such that $||f\mid_{[0,\pi]} - \phi||_\infty < \epsilon$. Let $0 < p_1 < p_2 < \ldots < p_{k-1} < \pi$ be a partition of $[0,\pi]$ compatible with ϕ. Let $c_i = \phi_{(p_{i-1},p_i)}$. Let ψ be a function on $[-\pi,\pi]$ defined as follows: $\psi(x) = \phi(x)$ for $x \geq 0$ and $\psi(x) = \phi(-x)$ for $x < 0$. Please check that $\psi \in S[-\pi,\pi]$. By construction, $-\pi < -p_{k-1} < -p_{k-2} < -p_{-1} < 0 < p_1 < \ldots < p_{k-1} < \pi$ is a partition of $[-\pi,\pi]$ compatible with ψ and $\psi\mid_{(-p_i,-p_{i-1})} = c_i$. Moreover, as $f(x) = f(-x)$,

$$||f - \psi||_\infty < \epsilon.$$

By definition,

$$\int_{-\pi}^{\pi} \psi = \sum_{i=1}^{k} \psi\mid_{(-p_i,-p_{i-1})} (p_i - p_{i-1}) + \sum_{i=1}^{k}\sum_{i=1}^{k} \psi\mid_{(p_{i-1},p_i)} (p_i - p_{i-1})$$

$$= 2\sum_{i=1}^{k} \phi\mid_{(p_{i-1},p_i)} (p_i - p_{i-1}) = 2\int_0^{\pi} \phi.$$

As $\epsilon > 0$ is arbitrary, and ψ and ϕ are ϵ-uniformly close to f, the derived equality implies that

$$\int_{-\pi}^{\pi} f = 2\int_0^{\pi} f.$$

18. This question is very simple yet its importance for asymptotic analysis of Fourier series is hard to overestimate. (i)

$$\int_p^q sint(x) == -\frac{cos(tx)}{t} \mid_p^q \to 0$$

as $t \to \infty$. (ii) Using a partition compatible with ϕ,

$$\int_a^b sin(tx)\phi(x) = \sum_{i=1}^{k} \phi\mid_{(p_{i-1},p_i)} \int_{p_{i-1}}^{p_i} sin(tx)dx \to 0,$$

as $t \to \infty$ as each integral under the sign of summation goes to zero by part (i). (iii) Fix $\epsilon > 0$. Let ϕ be a step function on $[a, b]$ such that $||f - \phi||_\infty < \epsilon$. Then

$$|\int_a^b sin(tx)(f(x) - \phi(x))dx| \leq \int_a^b |sin(tx)(f(x) - \phi(x))|dx \leq \epsilon(b - a).$$

Therefore, uisng (ii),

$$-\epsilon(b - a) \leq \lim_{t\to\infty} \int_a^b sin(tx)f(x) \leq \epsilon(b - a).$$

As ϵ is arbitrary, the above inequality leads to

$$\lim_{t\to\infty} \int_a^b sin(tx)f(x) = 0.$$

19. By definition,

$$D(\sum_{k=0}^{d} p_k x^k) = \sum_{k=1}^{d} k p_k x^{k-1}, \tag{6}$$

$$I(\sum_{k=0}^{d} p_k x^k) = \sum_{k=0}^{d} \frac{p_k}{k+1} x^{k+1}. \tag{7}$$

It is clear from these formulae that I, D are linear maps, alternatively one could quote the linearity of operations of differentiation and integration.

We can see from (6), that D is not injective: constants (polynomials of zero degree d) are mapped to zero. In fact, only constants are mapped to zero. Therefore $Ker(D)$ consists of the set of polynomials of zero degree which is isomophic to $\mathbb{R}$. D is surjective: the pre-image of an arbitrary polynomial $\sum_{k=0}^{d} q_k x_k$ is $\sum_{k=1}^{d+1} q_{k-1} \frac{x^k}{k}$.

It follows from (7) that I is not surjective: constants are not in the image of I. However, I is injective: if $I(p) = 0$, it is easy to see from (7), that $p = 0$.

Finally, $D \circ I \neq I \circ D$, as $D \circ I(1) = 1$, but $I \circ D(1) = 0$. In fact, as it is easy to check from (6, 7), $D \circ I = id$, whereas $I \circ D$ is a projection onto a linear subspace of $\mathbb{R}[x]$ consisting of polynomials which are equal to zero at $x = 0$, $I \circ D(\sum_{k=0}^{d} p_k x^k) = \sum_{k=1}^{d} p_k x^k$.

20. By the FTC and the continuity of f, the function $F(x) = \int_a^x f$, $x \in [a, b]$ is continuous on $[a, b]$ and differentiable on (a, b). Therefore, by the Mean Value Theorem, there is $c \in (a, b)$:

$$F(b) - F(a) = F'(c)(b - a) = f(c)(b - a).$$

We are done as $F(a) = 0$.