

15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to three randomly chosen 'B' and one 'A' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' questions.

The answers to ALL questions are to be submitted by the deadline of 3pm on Monday 26st October 2015. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeonloft if you are a Maths student, or in the dropbox labelled with the course's code, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

0.1 Step functions. Integration of step functions.

1. A. Construct a function $f : [0, 1] \rightarrow \mathbb{R}$ such that f is **not** a step function, but for any $\epsilon \in (0, 1)$ the restriction of f to $[\epsilon, 1]$ is a step function.
2. A. Show that, given any two step functions on the same interval, there is a partition compatible with both of them.
3. A. Prove that an arbitrary linear combination of step functions $\phi_1, \phi_2, \dots, \phi_n \in S[a, b]$ is a step function.
4. B. Is it possible to find a set $\{\phi_1, \phi_2, \dots, \phi_n\} \subset S[a, b]$ for some $n \in \mathbb{N}$ such that any step function on $[a, b]$ is representable as a linear combination of $\phi_1, \phi_2, \dots, \phi_n$? Justify your answer.
5. A. Let $\psi, \phi \in S[a, b]$. Prove that the product $\phi\psi$ is also a step function. ($S[a, b]$ is a commutative algebra over \mathbb{R} .)
6. B. Let $\psi : [0, \infty) \rightarrow \mathbb{R}$ be the integer part function. (By definition, for any $n \in \mathbb{N}$, $\psi|_{[n-1, n)} = n - 1$.) Sketch the graph of ψ . Show that for any $x > 0$, the restriction of ψ to $[0, x]$ is a step function. Define $\Psi : [0, \infty) \rightarrow \mathbb{R}$ as follows:

$$\Psi(x) = \int_0^x \psi(t) dt.$$

(i) Calculate $\Psi(x)$. Find the set of points $C \subset \mathbb{R}$ at which Ψ is continuous.

(ii) Find the set of points $D \subset \mathbb{R}$ at which Ψ is differentiable. For every $y \in D$ calculate $\Psi'(y)$.

7. A. Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a step function, and fix $K > 0$ in \mathbb{R} . Define $\psi : [Ka, Kb] \rightarrow \mathbb{R}$ by $\psi(x) := \varphi(x/K)$. Prove that ψ is a step function, and that $\int_{Ka}^{Kb} \psi = K \int_a^b \varphi$.
8. A. Let $\varphi, \psi : [a, b] \rightarrow \mathbb{R}$ be step functions. Prove that $|\varphi|$ is a step function and that $|\int_a^b \varphi| \leq \int_a^b |\varphi|$. Prove that $\int_a^b |\varphi + \psi| \leq \int_a^b |\varphi| + \int_a^b |\psi|$.
9. B. Prove that a continuous step function on $[a, b]$ is constant.
10. B. Let $F[a, b]$ be a vector space of functions defined on an interval $[a, b] \subset \mathbb{R}$. We assume that $F[a, b]$ is a 'nice' space, in the sense that (i) $F[a, b]$ contains all constant functions; (ii) $F[c, d]$ can be defined for any subinterval $[c, d] \subset [a, b]$ and if $f \in F[a, b]$ then the restriction of f to $[c, d]$ is in $F[c, d]$. For example, $F[a, b]$ can be the space of bounded or continuous or step functions. In general, integrals are often thought of as maps from $F[a, b]$ to \mathbb{R} . A map¹ $I_{[a, b]} : F[a, b] \rightarrow \mathbb{R}$ is called a 'good' integral if it satisfies the following properties:

(a) Linearity (for any $f, g \in F[a, b]$ and $\alpha, \beta \in \mathbb{R}$, $I_{[a, b]}(\alpha f + \beta g) = \alpha I_{[a, b]}(f) + \beta I_{[a, b]}(g)$)

(b) Monotonicity (if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $I_{[a, b]}(f) \geq I_{[a, b]}(g)$)

(c) Additivity (for any $f \in F[a, b]$ and any $q, p, r \in [a, b]$, $I_{[q, p]}(f) + I_{[p, r]}(f) = I_{[q, r]}(f)$)

(d) Compatibility with integrals of constant functions (If $c \in F[a, b]$ is a constant function, then $I_{[a, b]}(c) = c(b - a)$)

(e) Insensitivity to values on finite sets (If $f = 0$ at every point of $[a, b]$ except for a finite set then $I_{[a, b]}(f) = 0$)

(i) Check that the integral we defined on the space of step functions during the lectures is 'good'. (ii) Conversely, prove that the only map from $S[a, b]$ to \mathbb{R} which has five properties listed above is the integral defined in the lectures. **Hint.** This question is a good illustration of the rule stating that the more complicated the statement of the theorem is, the easier the proof.

¹More precisely, a family of maps labelled by subintervals of \mathbb{R} .

0.2 Regulated functions. Integration of regulated functions

11. A. If $\varphi : [0, 1] \rightarrow \mathbb{R}$ is a step function and $f : [0, 1] \rightarrow \mathbb{R}$ is given by $f(0) = 0, f(x) = \sin(1/x), 0 < x \leq 1$ show that $\|\varphi - f\|_\infty \geq 1$. Deduce that f is not regulated.
12. A. Prove that the collection of all regulated functions on $[a, b]$ is a vector space which contains the constant functions.
13. A. Prove that $f(x) = x$ is regulated on $[0, 1]$ by constructing an explicit sequence of step functions converging to f uniformly.
14. A. Give an example of a continuous function on $(0, 1]$ which is not regulated.
15. A. Show that any regulated function on $[a, b]$ is bounded.
16. B. Show that any monotonic function on $[a, b]$ is regulated.
17. B. Let $f : [a, b] \rightarrow \mathbb{R}$ be a regulated function and fix $K \neq 0$ in \mathbb{R} . Define $g : [Ka, Kb] \rightarrow \mathbb{R}$ by $g(x) := f(x/K)$. Prove that g is a regulated function, and that $\int_{Ka}^{Kb} g = K \int_a^b f$.
18. A. If the regulated function $\varphi : [-\pi, \pi] \rightarrow \mathbb{R}$ is odd (meaning that $\forall x \in [-\pi, \pi] \varphi(x) = -\varphi(-x)$) show that $\int_{-\pi}^{\pi} \varphi = 0$.
19. B. For $f : [1, 2] \rightarrow \mathbb{R}, f(x) := 1/x$ choose a sequence of step functions φ_n approximating f with partition $P_n := \{r/n : n \leq r \leq 2n\}$ to show that

$$\frac{1}{n+1} + \dots + \frac{1}{2n} \rightarrow \int_1^2 x^{-1} dx \text{ as } n \rightarrow \infty.$$

Calculate $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k}$. **Hint.** To calculate elementary integrals appearing in Questions 19 and 20 use your pre-university knowledge of integration.

20. B. (Riemann-Lebesgue lemma.) For $p < q$ in \mathbb{R} and $\alpha \in \mathbb{R}$ show that $\int_p^q \cos(tx + \alpha) dx \rightarrow 0$ as $t \rightarrow \infty$. For $\varphi \in S[a, b]$ show that $\int_a^b \varphi(x) \cdot \cos(tx + \alpha) dx \rightarrow 0$ as $t \rightarrow \infty$. [Suggestion: treat separately each of the intervals on which φ is constant.] For $f \in R[a, b]$ show that $\int_a^b f(x) \cdot \cos(tx + \alpha) dx \rightarrow 0$ as $t \rightarrow \infty$.

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