MA244 Analysis III **Solution Sheet 1.**

NB. THESE ARE SKELETON SOLUTIONS, **USE WISELY!**

0.1 Step functions. Integration of step functions.

1. Let us define f as follows: $f|_{\left(\frac{1}{n+1},\frac{1}{n}\right)} = (-1)^n$, $n \in \mathbb{N}$, $f(0) = 0$. **A.** $f \notin S[0,1]$: take any $p_0 \in (0,1)$. Choose a natural $n > \frac{1}{p_0}$. Then $\left(\frac{1}{n+1},\frac{1}{n}\right) \subset (0,p_0)$ and $\left(\frac{1}{n+2},\frac{1}{n+1}\right) \subset (0,p_0)$ This implies that $f|_{(0,p_0)}$ is not constant. As p_0 is arbitrary, we proved that there exists no partition of $[0,1]$ compatible with f. Therefore, $f \notin S[0,1]$. **B.** For any $\epsilon \in (0,1)$, $f \in S[\epsilon,1]$: the compatible partition is given by $\{\epsilon\} \cup ((\epsilon,1] \cap \{1,1/2,1/3,1/4,\ldots\})$.

2. Let $f,g \in S[a,b]$. Let P_f and P_g are partitions of $[a,b]$ compatible with f and g correspondingly. The the refined partition $P_f \cup P_g$ is compatible with both f and g.

3. The result follows by induction from the fact that for any $\phi,\psi \in S[a,b]$ and $\lambda,\mu \in \mathbb{R}$, $\lambda\phi+\mu\psi \in S[a,b]$. This fact is proved as follows: if P is a partition of $[a,b]$ compatible with ϕ and Q is a partition of $[a,b]$ compatible with ψ, then the refined partition $P \cup Q$ is compatible with $\lambda\phi+\mu\psi$.

4. No, it is impossible. Assume that we found such a set. Applying the conclusion of question 2 inductively, we can construct a partition of $[a,b]$ compatible with each member of this set. Therefore, there exists $p < q$: $[p,q] \in [a,b]$ and $\phi_k|_{[p,q]}$ is constant for each $k = 1,2,\ldots,n$. Define the following function f on $[a,b]$: $f(x) = 1$ for $x \in [a,(p+q)/2)$, $f(x) = -1$ for $x \in [(p+q)/2,b]$. Clearly, $f \in S[a,b]$, but f cannot be exressed as a linear combination of ϕ's as it is not constant on $[p,q]$.

5. Let P be a partition of $[a,b]$ compatible with both ϕ and ψ. Existence of such a partition was established in question 2. The function $\phi\psi$ is compatible with P: if $\psi|_{(p_i,p_{i+1})} = \psi_i$ and $\phi|_{(p_i,p_{i+1})} = \phi_i$ are constants, then $\psi\phi|_{(p_i,p_{i+1})} = \psi_i\phi_i$ is also a constant.

6. The sketch is ommited. ψ is a step function in $[0,x]$ due to the fact that $\psi\ |_{(n-1,n)}$ is constant for any $n \in \mathbb{N}$. (i) Using the additivity of the integral and for $n \in \mathbb{N}$,

$$\Psi(n) := \int_0^n \psi = \sum_{k=1}^n \psi|_{(k-1,k)} = \sum_{k=1}^n (k-1) = \frac{n(n-1)}{2}.$$

Any $x \in [0,\infty)$ can be represented in the form $x = \psi(x) + r$, where $r = x - \psi(x) \in [0,1)$. Therefore,

$$\Psi(x) = \Psi(\psi(x)) + \int_{\psi(x)}^{\psi(x)+r} \psi = \frac{\psi(x)(\psi(x)-1)}{2} + \psi(x)r = x\psi(x) - \frac{\psi(x)(\psi(x)+1)}{2},$$

where the penultimate equality is due to our result for $\Psi(n)$, and the last equality used the definition of r. ψ is continuous on $\mathbb{R} \setminus \{0,1,2,\ldots\}$ and right continuous at $\{0,1,2,\ldots\}$. Therefore, Ψ is right continuous on $[0,\infty)$ and left continuous on $\mathbb{R}\setminus\{0,1,2,\ldots\}$ (Analysis II). But

$$\begin{aligned}\lim_{x\to n-}\Psi(x) &= \psi(n-)+\frac{\psi(n-)(\psi(n-)-1)}{2}\\ &= n-1+\frac{(n-1)(n-2)}{2}=\frac{n(n-1)}{2}=\Psi(n).\end{aligned}$$

Therefore, Ψ is also left continuous at $\{1,2,3,\ldots\}$. Conclusion: $C=[0,\infty)$.

(ii) It clear from the representation of Ψ derived above that $D=\mathbb{R}\setminus\{1,2,\ldots\}$ (Ψ is right-differentiable at 0) and that for any $y\in D$, $\Psi'(y)=\psi(y)$.

7. If $\{a=p_0<p_1<\ldots<p_n<b=p_{n+1}\}$ is a partition of $[a,b]$ compatible with φ, then $\{Ka<Kp_1<\ldots<Kp_n<Kb\}$ is a partition of $[Ka,Kb]$ compatible with $\psi=\varphi(K^{-1}\cdot)$. Therefore, $\psi\in S[Ka,Kb]$. Let $\psi_i=\varphi_i$ be the value of ψ or φ on the i-th interval of the appropriate partition. By definition of the integral,

$$\int_{Ka}^{Kb}\psi=\sum_{i=1}^{n+1}\varphi_i(Kp_i-Kp_{i-1})=K\sum_{i=1}^{n+1}\varphi_i(p_i-p_{i-1})=\int_a^b\varphi$$

8. Let $P=\{a=p_0<p_1<\ldots<p_n<b=p_{n+1}\}$ be a partition compatible with both ψ,φ (see Question 2.) If $\varphi|_{(p_{i-1},p_i)}$ is a constant equal to φ_i, then $|\varphi|_{(p_{i-1},p_i)}$ is also a constant equal to $|\varphi_i|$. Therefore, P is compatible with $|\varphi|$ and $|\varphi|$ is a step function. Then

$$|\int_a^b\varphi|=|\sum_{i=1}^{n+1}\varphi_i(p_i-p_{i-1})|\le\sum_{i=1}^{n+1}|\varphi_i|(p_i-p_{i-1})=\int_a^b|\varphi|.$$

Similarly,

$$|\int_a^b(\varphi+\psi)|=|\sum_{i=1}^{n+1}(\varphi_i+\psi_i)(p_i-p_{i-1})|\le\sum_{i=1}^{n+1}(|\varphi_i|+|\psi_i|)(p_i-p_{i-1})=\int_a^b|\varphi|+\int_a^b|\psi|.$$

Here $\psi_i=\psi|_{(p_{i-1},p_i)}$.

9. Let f be our continuous step function on $[a,b]$, $P=\{a=p_0<p_1<\ldots<p_n<b=p_{n+1}\}$ - a partition of $[a,b]$ compatible with f and let $f_i=f|_{(p_{i-1},p_i)}$. The continuity requires that at every point c of $[a,b]$, $\lim_{x\to c-}f(x)=\lim_{x\to c+}f(x)=f(c)$. Applying this to $c=p_i$ gives $f_{i-1}=f_i=f(p_i)$ for $i=1,\ldots,n+1$. Finally, using one-sided continuity at the boundary points a and b we conclude that $f\equiv f_1$, i. e. constant.

10. Let $f,g\in S[a,b]$, let $P=\{a=p_0<p_1<\ldots<p_n<b=p_{n+1}\}$ - a partition of $[a,b]$ compatible with both f and g. Let $f_i=f|_{(p_{i-1},p_i)}$, $g_i=g|_{(p_{i-1},p_i)}$, $i=1,2,\ldots,n+1$.

   (a) Linearity:

$$\int_a^b(\alpha f+\beta g)=\sum_{i=1}^{n+1}(\alpha f_i+\beta g_i)(p_i-p_{i-1})$$

$$=\alpha\sum_{i=1}^{n+1}f_i(p_i-p_{i-1})+\beta\sum_{i=1}^{n+1}g_i(p_i-p_{i-1})=\alpha\int_a^b f+\beta\int_a^b g.$$

(b) Monotonicity: if $f \geq g$, then $f_i \geq g_i$ for every i. Therefore,

$$\int_a^b f - \int_a^b g = \sum_{i=1}^{n+1} (f_i - g_i)(p_i - p_{i-1}) \geq 0.$$

(c) Additivity: restrictions of step functions to subintervals are also step functions. Let $\{q = u_0 < u_1 < \ldots < u_n < p = u_{n+1}\}$ be a partition of $[q,p]$ compatible with $f|_{[q,p]}$ and let $\{p = v_0 < v_1 < \ldots < v_m < r = v_{m+1}\}$ be a partition of $[p,r]$ compatible with $f|_{[p,r]}$. Let $f|_{[u_{i-1},u_i]} = \alpha_i$ and $f|_{[v_{j-1},v_j]} = \beta_j$. Then the set $\{q = u_0 < u_1 < \ldots < u_n < v_0 < v_1 < \ldots < v_m < r = v_{m+1}\}$ is a partition of $[q,r]$ compatible with $f|_{[q,r]}$ and

$$\int_q^p f + \int_p^r f = \sum_{i=1}^{n+1} \alpha_i (u_i - u_{i-1}) + \sum_{i=1}^{m+1} \beta_i (v_i - v_{i-1}) = \int_q^r f.$$

(d) Compatibility with integrals of constants: if f is equal to constant c on $[a,b]$, then $\{a,b\}$ is a partition of $[a,b]$ compatible with f. Then by definition of the integral, $\int_a^b f = c(b-a)$.

(e) Insensitivity to values on finite sets: if f is zero on $[a,b] \setminus \{p_1, p_2, \ldots p_n\}$, then $f \in S[a,b]$ with a compatible partition $\{a = p_0 < p_1 < p_2 < \ldots < p_n < p_{n+1} = b\}$. (Here we assume that p_i's are internal points of $[a,b]$, all other cases can be studied in the similar vein.) But $f|_{(p_{i-1},p_i)} = 0$. Therefore,

$$\int_a^b f = \sum_{i=1}^{n+1} 0 \cdot (p_i - p_{i-1}) = 0.$$

0.2 Regulated functions. Integration of regulated functions.

11. (Q11) Let ψ be *any* step function on $[0,1]$. Let $P = \{0 < p_1 < p_2 < \ldots\}$ be a partition of $[0,1]$ compatible with ψ. Let $\psi_1 = \psi\mid_{(0,p_1)}$. Let $n \in \mathbb{N}$ be large enough so that $a_1 := \frac{1}{\pi/2+2\pi n} < p_1$ and $a_{-1} = \frac{1}{3\pi/2+2\pi n} < p_1$. Therefore, $a_1, a_2 \in (0,p_1)$. But $f(a_1) = 1$ and $f(a_2) = -1$. Therefore, $||f - \psi||_\infty \geq max(|\psi_1 - 1|, |\psi_1 + 1|) \geq 1$. Thus there exists no sequence of step functions converging to f uniformly meaning that f is not regulated.

12. (Q12). $R[a,b]$ contains all constant functions as $S[a,b] \subset R[a,b]$ and a constant function is a step function. (1) $R[a,b]$ is closed w. r. t. multiplication by numbers: let $r \in R[a,b], \alpha \in \mathbb{R}$. For any $\epsilon > 0$ there exists a step function ψ_ϵ: $||r - \psi_\epsilon||_\infty < \epsilon$. Therefore, $||\alpha r - \alpha\psi_\epsilon||_\infty < \epsilon|\alpha|$. As $\alpha\psi_\epsilon \in S[a,b]$ and ϵ is arbitrary positive, we conclude that there is a sequence of step functions converging uniformly to αr. Therefore, αr is regulated. (2) $R[a,b]$ is closed under addition: let p, q be regulated. Then for any $\epsilon > 0$ there exist $\phi_\epsilon, \psi_\epsilon \in S[a,b]$: $sup_{x\in[a,b]}|p(x) - \phi_\epsilon(x)| < \epsilon/2$, $sup_{x\in[a,b]}|q(x) - \psi_\epsilon(x)| < \epsilon/2$. So,

$$sup_{x\in[a,b]}|p(x) + q(x) - \phi_\epsilon(x) - \psi_\epsilon(x)|$$
$$\leq sup_{x\in[a,b]}(|p(x) - \phi_\epsilon(x)| + |q(x) - \psi_\epsilon(x)|)$$
$$\leq sup_{x\in[a,b]}|p(x) - \phi_\epsilon(x)| + sup_{y\in[a,b]}|q(y) - \psi_\epsilon(y)| < \epsilon.$$

As $\psi_\epsilon + \phi_\epsilon \in S[a,b]$, we conclude that $p + q$ is regulated.

13. (Q13) For any $n \in \mathbb{N}$, let $\phi_n(x) = \frac{1}{n} integer(nx)$, where *integer* is the integer part function on $[0, \infty)$. For any $x \geq 0$, $0 \leq nx - integer(nx) \leq 1$. Therefore, $||f - \phi_n||_\infty \leq \frac{1}{n} \to 0$ as $n \to \infty$.

14. (Q14) Example: $f(x) = 1/x$: f is continuous on $(0, 1]$ (Analysis II). Let ψ be any step function on $(0, 1]$ and let $\psi_1 = \psi \mid_{(0,p_1)}$, where $\{p_1 < p_2 < \ldots < p_n = 1\}$ is a partition of $(0, 1]$ compatible with ψ. Let $\epsilon \in (0, p_1)$. Then $||f - \psi||_\infty \geq |1/\epsilon - \psi_1| \to \infty$ as $\epsilon \to 0$. Therefore, there exists no sequence of step functions on $(0, 1]$ converging to f uniformly.

15. (Q15) As $f \in R[a, b]$, there is $\psi \in S[a, b]$: such that for any $x \in [a, b]$, $\psi(x) - 1 \leq f(x) \leq \psi(x) + 1$. Any step function is bounded: the upper (the lower) bound is given by the maximum M (the minimum m) of the finite set of values of the step function over $[a, b]$. Therefore, for any $x \in [a, b]$, $|f(x)| \leq max(|M| + 1, |m| + 1)$, i. e. $f \in B[a, b]$.

16. (Q16) A monotonic function is either non-decreasing or non-increasing. Let $f : [a, b] \to \mathbb{R}$ be non-decreasing (the second case can be analysed in the same way). To prove that f is regulated we will construct, for any $n \in \mathbb{N}$, a function $\varphi \in S[a.b]$: $||f - \phi||_\infty < 1/n$. $f(a) \leq f(b)$ so pick $r \in \mathbb{Z}, k \in \mathbb{N}$ with $r/n \leq f(a) < (r + 1)/n$ and $(r + k - 1)/n < f(b) \leq (r + k)/n$. For $0 < j \leq k$ put $p_j := \sup\{x \in [a, b] : f(x) \leq (r + j)/n\}$ and put $p_0 := a$. Then $x \in (p_{j-1}, p_j) \Rightarrow (r + j - 1)/n < f(x) \leq (r + j)/n$. (If $p_{j-1} = p_j$ then $((r + j - 1)/n, (r + j)/n) \cap f([a, b]) = \emptyset$ and we should renumber $P = \{p_0, \ldots, p_k\}$.) Put $\varphi = (r + j)/n$ on (p_{j-1}, p_j) and $\varphi(p_j) := f(p_j)$. Then $\forall j \in \{1, \ldots, k\}$ $\forall x \in (p_{j-1}, p_j)$ $|\varphi(x) - f(x)| = |(r + j)/n - f(x)| \leq 1/n$ (and also for $x = p_j$) so $\|\varphi - f\|_\infty \leq 1/n$. Thus f is regulated.

17. (Q17) Let $\{\phi_n\}_{n\geq 1}$ be a sequence of step functions on $[a, b]$ converging to f uniformly ($a < b$). Then $\{\phi_n(K^{-1}\cdot)\}_{n\ \geq 1}$ is a sequence of step functions on $[min(Ka, Kb), max(Ka, Kb)]$ converging to $f(K^{-1}\cdot)$ uniformly. Therefore, $f(K^{-1}\cdot)$ is regulated. Let $P = \{a = p_0 < p_1 < \ldots < p_n < b = p_{n+1}\}$ be a partition of $[a, b]$ compatible with ϕ_n. Then $K \cdot P$ is a partition of $[min(Ka, Kb), max(Ka, Kb)]$ compatible with $\phi_n(K\cdot)$. It is an immediate consequence of the definition of the step integral that

$$\int_{min(Ka,Kb)}^{max(Ka,Kb)} \phi_n(K^{-1}\cdot) = |K| \int_a^b \phi_n.$$

Taking the large-n limit of both sides we find

$$\int_{min(Ka,Kb)}^{max(Ka,Kb)} f(K^{-1}\cdot) = |K| \int_a^b f.$$

Using the convention $\int_c^d h = -\int_d^c h$ we find that

$$\int_{min(Ka,Kb)}^{max(Ka,Kb)} h = sign(K) \int_{Ka}^{Kb} h.$$

Combining the last two formulae, we find that

$$\int_{Ka}^{Kb} f(K^{-1}\cdot) = K \int_a^b f.$$

18. (Q18) The formula of Question 17 specified to the case $K = -1$ states

$$\int_{\pi}^{-\pi} \varphi(-x)dx = -\int_{-\pi}^{\pi} \varphi(x)dx.$$

Using $\phi(-x) = -\phi(x)$ and swapping the order of limits in the l. h. s. of the above formula, we find that $\int_{-\pi}^{\pi} \phi = -\int_{-\pi}^{\pi} \phi$, which implies the desired result.

19. (Q19) Let φ_n: $\varphi_n\,|_{(r/n,(r+1)/n]} = n/(r+1)$, $r = n, n+1, \ldots, 2n-1$. $\varphi_n(1) = 1$. Then $||f - \varphi_n||_\infty \le max_{n \le r \le 2n}|n/r - n/(r+1)| \le 1/n \to 0$ as $n \to \infty$. Therefore, $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of step functions converging to f uniformly. Thefore $\int_1^2 \phi_n \to \int_1^2 f$ by definition of the regulated integral. So,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \int_1^2 \frac{1}{x}dx = log(2).$$

20. (Q20) (i) $|\int_p^q \cos(tx+\alpha)dx| = |t^{-1}[\sin(tx+\alpha)]_p^q| = t^{-1}|\sin(tp+\alpha) - \sin(tq+\alpha)| \le 2t^{-1} \to 0$ as $t \to \infty$.

    (ii) Let $\varphi \in S[a,b]$ and let $\{p_0 < p_1 \ldots < p_k\}$ be a compatible partition of [a,b]. Then $|\int_a^b \varphi(x)\cos(tx+\alpha)\,dx| = |\sum_{j=1}^k c_j \int_{p_{j-1}}^{p_j} \cos(tx+\alpha)\,dx| \le \sum_{j=1}^k |c_j| \cdot 2t^{-1} \to 0$ as $t \to \infty$.

    (iii) Given $f \in R[a,b]$ and $\varepsilon > 0$ choose $\varphi \in S[a,b]$ with $\|f - \varphi\|_\infty \le \varepsilon$. Then $\sup_x\{|f(x)\cdot\cos(tx+\alpha) - \varphi(x)\cdot\cos(tx+\alpha)|\} \le \|f-\varphi\|_\infty \le \varepsilon$.
    By the last part $\exists\, T$ such that $t \ge T \Rightarrow |\int_a^b \varphi(x)\cdot\cos(tx+\alpha)\,dx| < \varepsilon$, so $t \ge T \Rightarrow |\int_a^b f(x)\cdot\cos(tx+\alpha)\,dx| \le |\int_a^b (f(x)\cdot\cos(tx+\alpha) - \varphi(x)\cdot\cos(tx+\alpha))\,dx| + |\int_a^b \varphi(x)\cdot\cos(tx+\alpha)\,dx| < \varepsilon(b-a) + \varepsilon = \varepsilon(b-a+1)$. Hence $\int_a^b f(x).\cos(tx+\alpha)\,dx \to 0$ as $t \to \infty$.

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