

15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to two randomly chosen 'B' and two 'A' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' questions.

The answers to ALL questions are to be submitted by the deadline of 3pm on Monday, the 3rd of November 2014. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeonloft if you are a Maths student, or in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

0.1 Integration

1. A. Give an example of a regulated function $f : [a, b] \rightarrow \mathbb{R}$ with the properties that $\forall x \in [a, b] f(x) \geq 0$, $\int_a^b f = 0$ and there is $c \in [a, b]$ with $f(c) > 0$. **For glory (but not for credit):** Find f satisfying all of the above, which is NOT a step function.
2. A. Show that a continuous function $f : [a, b] \rightarrow \mathbb{R}$ with the properties that $\forall x \in [a, b] f(x) \geq 0$ and $\int_a^b f = 0$ must be identically zero.
3. A. Find the points of the extremum of the function

$$f(x) = \int_1^x \frac{\sin(t)}{t} dt$$

in the region $x > 1$.

4. A. Find the derivatives of the following functions:

(a) $F(x) = \int_1^x \log(t) dt$, $x > 1$;

(b) $G(x) = \int_x^0 \sqrt{1+t^4} dt$, $x \in \mathbb{R}$;

(c) $H(x) = \int_x^{x^2} e^{-t^2} dt$, $x \in \mathbb{R}$;

(d) $I(x) = \int_{\frac{1}{x}}^{\sqrt{x}} \cos(t^2) dt$, $x > 0$.

Hint. Use the chain rule, the FTC and the following two facts: $\int_a^b f = -\int_b^a f$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

5. B. Applying the second version of the fundamental theorem of calculus (aka Newton-Leibnitz formula) find the following integrals:

(a) $\int_0^1 \log(1+x) dx$;

(b) $\int_{-2}^{-1} \frac{1}{x^3} dx$;

- (c) $\int_{-x}^x e^t dt$, $x \in \mathbb{R}$;
 (d) $\int_0^x t \cdot \cos(t^2) dt$, $x > 0$.

0.2 Properties of regulated functions.

6. B. Define $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) := 0$ if $x \notin \mathbb{Q}$, $f(p/q) := 1/q$, $q > 0$, p, q coprime integers.
 (i) Show that f is continuous at $1/\sqrt{2}$ but not at $1/3$.
 (ii) Prove that $f \notin S[0, 1]$.
 (iii) Prove that f is regulated.
Hint. For any $\varepsilon > 0$ construct a step function $\varphi : [0, 1] \rightarrow \mathbb{R}$ such that $\|f - \varphi\|_\infty < \varepsilon$. Restrict to $q < 1/\varepsilon$.
7. B. Let $f : [0, 2] \rightarrow \mathbb{R}$ be a regulated function. Let $(x_n)_{n \geq 1}$ be a sequence in $[0, 1)$ with $\lim_{n \rightarrow \infty} x_n = 1$. Prove that the sequence $(f(x_n))_{n \geq 1}$ is Cauchy.

0.3 Improper integrals

8. A. Calculate $\lim_{R \rightarrow \infty} \int_{-R}^R x dx$. Does your result imply that $\int_{-\infty}^{\infty} x dx$ exists? Explain your answer.
9. B. Evaluate the improper integrals (or establish their divergence):
 (a) $\int_0^1 \log(t) dt$;
 (b) $\int_0^1 \frac{1}{t^p} dt$, $p > 0$;
 (c) $\int_1^\infty \frac{1}{t^p} dt$, $p > 0$;
 (d) $\int_0^\infty \cos(x) dx$.

0.4 Uniform convergence and uniform continuity.

10. A. Let $A \subset \mathbb{R}$ and let $f, g : A \rightarrow \mathbb{R}$ be uniformly continuous. Prove that $f + g : A \rightarrow \mathbb{R}$ is uniformly continuous.
11. B. If $a < b$ and $f : (a, b) \rightarrow \mathbb{R}$ is uniformly continuous, show that f is bounded. Give examples of functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ which are uniformly continuous such that f is not bounded but g is bounded.
12. B. If f is regulated and is the uniform limit of weakly increasing step functions show that f is weakly increasing. [f is weakly increasing means $u < v \Rightarrow f(u) \leq f(v)$.]