Questions for credit: 2 (4 points), 4 (5 points), 6 (8 points) and 9 (8 points)

0.1 Integration.

1. Let $f((a+b)/2) = 1$, $f(x) = 0$, $x \in [a,b] \setminus \{(a+b)/2\}$. f is a step function, therefore it is regulated; for any $x \in [a,b]$ $f(x) \geq 0$, $f((a+b)/2) > 0$ and $\int_a^b f = 0$.

2. If $f(c) > 0$ for some $c \in [a,b]$ then $\frac{1}{2}f(c) > 0$ so there is $\delta > 0$ such that $x \in (c-\delta, c+\delta) \cap [a,b] \Rightarrow |f(x) - f(c)| < \frac{1}{2}f(c) \Rightarrow f(x) > \frac{1}{2}f(c) \Rightarrow \forall x \in [a,b]$ $f(x) \geq \varphi(x)$ where the step function $\varphi : [a,b] \to \mathbb{R}$ is given by $\varphi(x) := \frac{1}{2}f(c)$ if $x \in (c-\delta, c+\delta) \cap [a,b]$ and $\varphi(x) = 0$ otherwise. Then, by the bounds on the integral of regulated functions, $\int_a^b f \geq \int_a^b \varphi = \delta f(c) > 0$ (but adjust $\int_a^b \varphi$ if $\min\{c-a, b-c\} < \delta$). Thus $\forall c \in [a,b]$ $f(c) = 0$.

3. The integrand is continuous in $[1,x]$. Therefore the integral defines a differentiable function. The derivative at the extremum points is zero. Applying FTC,
$$f'(x) = \frac{sin(x)}{x} = 0.$$
Therefore, $x = \pi \cdot k$, $k = 1, 2, \ldots$.

4. The hint leads to the following formula:
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f = b'(x)f(b(x)) - a'(x)f(a(x)).$$
Its application gives:
   (a) $F'(x) = log(x)$.
   (b) $G'(x) = -\sqrt{1+x^4}$.
   (c) $H'(x) = 2xe^{-x^4} - e^{-x^2}$
   (d) $I'(x) = \frac{1}{2}x^{-1/2}cos(x) + \frac{1}{x^2}cos(1/x^2)$.

5. All the integrals are straightforward, the first one requires integration by parts, the fourth - a substitution $y = t^2$.
   (a) $\int_0^1 log(1+x)dx = (xlog(x) - x) \mid_1^2 = 2log(2) - 1$;
   (b) $\int_{-2}^{-1} \frac{1}{x^3}dx = -1/2x^{-2}|_{-2}^{-1} = -\frac{3}{8}$;
   (c) $\int_{-x}^{x} e^t dt = e^x - e^{-x} = 2sinh(x)$;
   (d) $\int_0^x t \cdot cos(t^2)dt = \frac{1}{2}\int_0^{x^2} cos(y)dy = \frac{1}{2}sin(x^2)$.

0.2 Properties of regulated functions.

6. (i) Take a sequence of irrationals $x_n \to 1/3$; then $f(x_n) = 0 \to 0 \neq 1/3 = f(1/3) = f(\lim_{n\to\infty} x_n)$ so f is not continuous at $1/3$. As $c = 1/\sqrt{2}$ is irrational, $f(c) = 0$. For any fixed $\varepsilon > 0$, only finitely many rational numbers p/q have

$q \le 1/\varepsilon$, so let δ be the distance from c to the nearest of these. Then for any rational $x \in (c-\delta, c+\delta)$ and $x = p/q$, $0 < f(x) = 1/q < \varepsilon$; for any irrational $x \in (c-\delta, c+\delta)$, $f(x) = 0$. The last two facts imply that f is continuous at c.
(ii) Every open interval in $(0,1)$ contains irrational points x where $f(x) = 0$ and rational x where $f(x) \neq 0$, so it is not constant there and f cannot be a step function.
(iii) Given $\varepsilon > 0$, define $\varphi : [0,1] \to \mathbb{R}$ by $\varphi(p/q) := 1/q$ if $0 \le p \le q \in \mathbb{N}$ and p, q are coprime and $q \le 1/\varepsilon$. For all other $x \in [0,1]$ set $\phi(x) = 0$. Clearly, $\varphi \in S[0,1]$. Then $f(t) = \varphi(t)$ unless $t = p/q, q > 1/\varepsilon$. Hence $\|f - \varphi\|_\infty = \sup_{p/q \in [0,1]\cap\mathbb{Q}: q > 1/\varepsilon} |f(p/q)| = \sup_{p/q \in [0,1]\cap\mathbb{Q}: q>1/\varepsilon} \frac{1}{q} < \varepsilon$.

7. Given $\varepsilon > 0$ take a step function $\varphi[0,2] \to \mathbb{R}$ such that $\|\varphi - f\|_\infty < \varepsilon/2$. Take a partition P compatible with φ and include 1 in P. Then, for some j, $0 = p_0 < p_1 < \ldots < p_j = 1 < \ldots < p_k = 2$ and $\forall x \in (p_{j-1}, 1)$ $\varphi(x) = c_j$. Choose N such that $\forall n \ge N$ $x_n \in (p_{j-1}, 1)$. Then $m \ge n \ge N \Rightarrow |f(x_m) - f(x_n)| = |f(x_m) - \varphi(x_m) + \varphi(x_n) - f(x_n)| \le |f(x_m) - \varphi(x_m)| + |\varphi(x_n) - f(x_n)| \le \|f - \varphi\|_\infty + \|\varphi - f\|_\infty < 2\varepsilon/2 = \varepsilon$. Hence $(f(x_n))$ is Cauchy and so converges.

0.3 Improper integrals.

8. $\int_{-R}^{R} x dx = \frac{x^2}{2} |_{-R}^{R} = 0$. However, $\int_{-\infty}^{\infty} x dx$ is divergent, as

$$\lim_{R_1 \to \infty} \int_{-R_1}^{0} x dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} x dx$$

does not exist.

9. In all cases, the relevant integrals are elementary, in (a) use integration by parts to find the anti-derivative of $\log(t)$.

   (a) $\int_0^1 \log(t) dt = \lim_{\epsilon \downarrow 0} \int_\epsilon^1 \log(t) dt = \lim_{\epsilon \downarrow 0} (t \log(t) - t) |_\epsilon^1 = -1$. Here we used $\lim_{\epsilon \to 0} \epsilon \log(\epsilon) = 0$, which follows from l'Hôpital's rule.

   (b) There are three cases to consider: $0 < p < 1$, $p = 1$ and $p > 1$. If $0 < p < 1$, $\int_0^1 \frac{1}{t^p} dt = \lim_{\epsilon \downarrow 0} \int_\epsilon^1 \frac{1}{t^p} dt = \lim_{\epsilon \downarrow 0} \frac{t^{1-p}}{1-p} |_\epsilon^1 = \frac{1}{1-p}$. The improper integral exists and equals to $\frac{1}{1-p}$. If $p = 1$, $\int_0^1 \frac{1}{t} dt = \lim_{\epsilon \downarrow 0} \int_\epsilon^1 \frac{1}{t} dt = \lim_{\epsilon \downarrow 0} \log(t)|_\epsilon^1$. The limit does not exist, so the improper integral diverges. If $p > 1$, $\int_0^1 \frac{1}{t^p} dt = \lim_{\epsilon \downarrow 0} \int_\epsilon^1 \frac{1}{t^p} dt = \lim_{\epsilon \downarrow 0} \frac{t^{1-p}}{1-p} |_\epsilon^1$. The limit does not exist, so the improper integral diverges.

   (c) There are three cases to consider: $p > 1$, $p = 1$ and $0 < p < 1$. If $p > 1$, $\int_1^\infty \frac{1}{t^p} dt = \lim_{R \to \infty} \int_1^R \frac{1}{t^p} dt = \lim_{\mathbb{R} \to \infty} \frac{t^{1-p}}{1-p} |_1^R = \frac{1}{p-1}$. The improper integral exists and equals to $\frac{1}{p-1}$. If $p = 1$, $\int_0^\infty \frac{1}{t} dt = \lim_{R \to \infty} \int_1^R \frac{1}{t} dt = \lim_{R \to \infty} \log(t)|_1^\infty$. The limit does not exist, so the improper integral diverges. If $0 < p < 1$, $\int_1^\infty \frac{1}{t^p} dt = \lim_{R \to \infty} \int_1^R \frac{1}{t^p} dt = \lim_{R \to \infty} \frac{t^{1-p}}{1-p} |_1^\infty$. The limit does not exist, so the improper integral diverges.

   (d) $\int_0^\infty \cos(x) dx = \lim_{R \to \infty} \int_0^R \cos(x) dx = \lim_{R \to \infty} \sin(R)$. The limit does not exist, the improper integral is divergent.

0.4 Uniform convergence and uniform continuity.

10. As f, g are uniformly continuous on A, for any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $\forall x, y \in A$,
$$|x - y| < \delta_1 \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2},$$
$$|x - y| < \delta_2 \Rightarrow |g(x) - g(y)| < \frac{\epsilon}{2}.$$
Let $\delta = min(\delta_1, \delta_2) > 0$. Then for any $x, y \in A : |x - y| < \delta$
$$|f(x) + g(x) - f(y) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$
So, $f + g$ is uniformly continuous on A.

11. Take $\delta > 0$ such that $x, y \in (a, b)$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < 1$. Now take $n \in \mathbb{N}$ such that $n > (b - a)/\delta$ and put $p_j := a + j(b - a)/n$, $0 \leq j \leq n$. Then $\forall x \in (a, b)$ $\exists j \in \{1, \ldots, n\}$ such that $x \in [p_{j-1}, p_j]$. As $|p_j - p_{j-1}| < \delta$, it follows that $|f(x)| < |f(\frac{1}{2}(p_j + p_{j-1}))| + 1$. Hence $\sup\{|f(x)| : x \in (a, b)\} \leq \max\{|f(\frac{1}{2}(p_j + p_{j-1}))| + 1 : 1 \leq j \leq n\}$, which shows that f is bounded.
$f : \mathbb{R} \to \mathbb{R},\ f(x) := x$ is uniformly continuous (with $\delta = \varepsilon$) but not bounded. $g : \mathbb{R} \to \mathbb{R}, g := \sin$ is uniformly continuous (with $\delta = \varepsilon$, use $sin(x) - sin(y) = 2cos(\frac{x+y}{2})sin(\frac{x-y}{2})$) and bounded. A constant function g is a simpler example.

12. Suppose that f is the uniform limit of weakly increasing step functions and yet not itself weakly increasing. Then there are $u < v$ in $[a, b]$ with $f(u) > f(v)$. Put $\varepsilon = (f(u) - f(v))/3 > 0$ and pick a weakly increasing step function $\varphi : [a, b] \to \mathbb{R}$ with $\|\varphi - f\|_\infty \leq \varepsilon$. Then $f(u) - \varepsilon \leq \varphi(u) \leq \varphi(v) \leq f(v) + \varepsilon$ so $3\varepsilon = f(u) - f(v) \leq 2\varepsilon$ contradicting $\varepsilon > 0$.