15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to two randomly chosen 'B' and two 'A' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' questions.

The answers to ALL questions are to be submitted by the deadline of 3pm on Monday, the 9th of November 2015. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeonloft if you are a Maths student, or in the dropbox labelled with the course's code, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

0.1 Properties of regulated functions and their integrals.

1. *A. Prove that for any $f, g \in R[a,b]$, $f+g \in R[a,b]$, $f \cdot g \in R[a,b]$. ($R[a,b]$ is closed under addition and multiplication.)*

2. *A. Use the Riemann-Lebesgue lemma (see Q.20 of the first assignment) to calculate*

$$\lim_{t\to\infty} \int_a^b \sin^2(tx)\phi(x)dx,$$

   where $\phi \in R[a,b]$.

3. *B. The function $f(x) = x\sin(1/x)$ ($f(0) = 0$) is continuous hence regulated in $I = [0,1]$ and the function $g(x) = sign(x)$ ($g(x)=1$ if $x > 0$, $g(0) = 0$, $g(x) = -1$ if $x < 0$) is regulated in $[-1,1]$. Prove that the composed function $g \circ f$ is* **NOT** *regulated in I. (Recall that $g \circ f(x) := g(f(x))$.)*

4. *A. Give an example of a non-negative regulated function $f : [a,b] \to \mathbb{R}$ such that f is not identically zero on $[a,b]$, but $\int_a^b f = 0$. Is there a function satisfying all of the above which is not a step function?*

5. *B. Prove that the set of discontinuities of a regulated function on $[a,b]$ is countable.* **Hint.** *Use the following theorem which we will prove in the lectures: if $f \in R[a,b]$, then for any $x \in (a,b)$, the one-sided limit $\lim_{y\to x+} f(y)$ and $\lim_{y\to x-} f(y)$ exist. In addition the limits $\lim_{y\to a+} f(y)$ and $\lim_{y\to b-} f(y)$ exist.*

0.2 Integration

6. *B. (What did Newton and Leibnitz do for us?) Evaluate $\int_1^5 x^2 dx$ by finding a sequence of step functions on $[1,5]$ converging uniformly to $f(x) = x^2$ and calculating the limit of the corresponding sequence of step integrals.*

7. A. Let $I = \int_a^b \frac{1}{log(x)}dx$, $b > a > 1$. Find (i)$\frac{dI}{da}$; (ii) $\frac{dI}{db}$.

8. A. Find $I'(x)$, where $I = \int_{a(x)}^{b(x)} \frac{1}{log(t)}dt$, where a, b are differentiable functions on $\mathbb{R}$ with values in $[2, \infty]$.

9. B. Find the derivatives of the following functions:

   (a) $F(x) = \int_1^{exp(x)} log^{10}(t)dt,\ x > 0$;

   (b) $G(x) = \int_{x^2}^{1+x^2} \sqrt{1+t^4}dt,\ x \in \mathbb{R}$;

   (c) $H(x) = \int_{-x^2}^{x^2} e^{-t^5}dt,\ x \in \mathbb{R}$;

   (d) $I(x) = \int_{\frac{1}{x}}^{\sqrt{x}} cos(t^4)dt,\ x > 0$.

   Hint. *Use the chain rule, the FTC and the following two facts:* $\int_a^b f = -\int_b^a f$ *and* $\int_a^b f = \int_a^c f + \int_c^b f$.

10. B. Applying the second version of the fundamental theorem of calculus (aka Newton-Leibnitz formula) or any other method, find the following integrals:

    (a) $\int_1^e \frac{sin(log(x))}{x}dx$;

    (b) $\int_{log(2)}^{log(3)} \frac{1}{\cosh^2(x)}dx$;

    (c) $\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}dx$;

    (d) $\int_e^{e^2} \frac{1}{x\log(x)}dx$.

    (e) $\int_{-1}^1 x^{1001}e^{-x^{100}}dx$

0.3 Improper integrals

11. B. Prove the following comparison test: let f, F be continuous on (a, b). If the improper integral $\int_a^b F(x)dx$ coverges and $|f(x)| \le F(x)$ for all $x : a < x < b$, then the improper integral $\int_a^b f(x)$ converges as well. Here $-\infty \le a < b \le \infty$. Prove the test just for one particular case: $b = \infty$ and f, F are continuous on $[a, \infty)$. All other cases can be treated in a similar way.

12. A. Test the convergence of the integral $\int_0^\infty e^{-x^3}dx$.

13. A. Test the convergence of the elliptic integral $\int_0^1 \frac{1}{\sqrt{1-x^4}}dx$.

14. A. Calculate $\lim_{R\to\infty} \int_{-R}^R x^3dx$. Does your result imply that $\int_{-\infty}^\infty x^3dx$ exists? Explain your answer.

15. B. Test the convergence of the following integrals:

    (a) $\int_0^{100} \frac{1}{x^{1/3}+2x^{1/4}+x^3}$;

    (b) $\int_0^\infty \frac{sin(x)}{x^2}$.

0.4 Uniform convergence.

16. *A. Proceeding from the definition of uniform convergence, prove that the geometric series* $\sum_{k=0}^{\infty} x^k$ *does not converge uniformly in the interval* $(-1, 1)$ *but converges uniformly on any closed subinterval within this interval.*

17. *A. Using the definition of uniform convergence prove that the exponential series* $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ *converges uniformly on any finite subinterval of* $\mathbb{R}$*.*

18. *A. Prove the following criterion for unform convergence: a functional series* $\{g_n\}_{n\geq 1}$ *on* $A \subset \mathbb{R}$ *converges to function* g *uniformly iff* $\lim_{n\to\infty} \sup_{x\in A} |g_n(x) - g(x)| = 0$*.*

19. *A. Prove that the series*

$$(x^2 - x^4) + (x^4 - x^6) + \ldots + (x^{2n} - x^{2n+2}) + \ldots$$

converges pointwise on $[-1, 1]$*, but it does not converge on* $(-1, 1)$ *uniformly.*

20. *B. In general, pointwise convergence does not imply uniform convergence. Prove the following theorem: Let* $\{F_n\}_{n\geq 1}$ *be a sequence of continuous functions on* $[a, b]$*. Suppose that for all* $x \in [a, b]$ *and for all* $n \in \mathbb{N}$*,*

$$F_n(x) \geq F_{n+1}(x).$$

Let F *be a continuous function on* $[a, b]$*. Prove that the pointwise convergence of* F_n *to* F *implies the uniform convergence.*

26th of October 2015 *Sergey Nazarenko and Oleg Zaboronski*