NB. THESE ARE SKELETON SOLUTIONS, **USE WISELY !**

0.1 Properties of regulated functions and their Integrals.

1. (Q.1) Pick any $\epsilon > 0$. As f, g are regulated, there exist $\phi, \psi \in S[a,b]$: $||f - \phi||_\infty < \epsilon/2$, $||g - \psi||_\infty < \epsilon/2$. Notice that $\psi + \phi \in S[a,b]$ (Assignment 1). But
$$||f + g - \phi - \psi||_\infty \leq ||f - \phi||_\infty + ||g - \psi||_\infty < \epsilon.$$
Therefore $f + g \in R[a,b]$.

   To show that $fg \in R[a,b]$, notice that regulated functions are bounded. (Assignment 1.) Therefore there exist $M, N > 0$: $||f||_\infty < M$, $||g||_\infty < N$. Let $\phi, \psi \in S[a,b]$: $||f - \phi||_\infty < \delta/M$, $||g - \psi||_\infty < \delta/N$, Notice that $\phi\psi \in S[a,b]$. (Assignment 1.)

   Then
$$\begin{aligned} ||fg - \phi\psi||_\infty &= ||f(g - \psi) + g(f - \phi) + (f - \phi)(\psi - g)||_\infty \\ &\leq ||f(g - \psi)||_\infty + ||(f - \phi)g||_\infty + ||(f - \phi)(\psi - g)||_\infty \\ &\leq 2\delta + \delta^2/(MN). \end{aligned}$$
For any $\epsilon > 0$ the equation $2\delta + \delta^2/(MN) = \epsilon$ has a positive solution and we are done. The last inequality above uses the following property of the sup-norm:
$$||ab||_\infty = \sup_x |a(x)||b(x)| \leq \sup_x |a(x)| \sup_y |b(y)| = ||a||_\infty ||b||_\infty$$

2. (Q.2) $\sin^2(t\cdot)$ is continuous hence regulated on $[a,b]$. By Question 1, $\sin^2(t\cdot)\phi \in R[a,b]$, so the question is well posed. Recall that $\sin^2(tx) = \frac{1-\cos(2tx)}{2}$. By the RL lemma, $\int_a^b \cos(2tx)\phi(x)dx \to 0$ as $t \to \infty$. Therefore,
$$\lim_{t\to\infty} \int_a^b \sin^2(tx)\phi(x)dx = \frac{1}{2}\int_a^b \phi(x)dx.$$

3. (Q.3) Let $h := g \circ f$. Then $h(0) = 0$ and $h(x) = sign(x\sin(1/x)) = sign(\sin(1/x))$ for $x > 0$. So, $h(x) = 1$ for $x \in (\frac{1}{2\pi(n+1/2)}, \frac{1}{2\pi n})$ and $h(x) = -1$ for $x \in (\frac{1}{2\pi n}, \frac{1}{2\pi(n+1/2)})$. Such a function is not regulated as for any step function ϕ in I, $||h - \phi||_\infty > min_{c\in\mathbb{R}}(|1 - c|, |1 + c|) \geq 1$.

4. (Q. 4) Let f be defined as follows: $f(a + (b - a)/n) = 1/n$, $n \in \mathbb{N}$; $f(x) = 0$ if $x \notin a + (b-a)/\mathbb{N}$. Note that $f \in R[a,b]$: consider $\phi_k \in S[a,b] : \phi_k(x) = f(x)$ for $x \geq a + (b-a)/k$ and $\phi_k(x) = 0$ for $x < a + (b-a)/k$. Then $||f - \phi_k||_\infty \leq 1/k \to 0$ as $k \to \infty$. Therefore, the sequence of step functions $\{\phi_k\}$ converges to f uniformly and f is regulated by definition. Of course, f itself is not a step function as the cardinality of its range is infinite. The integral:
$$\int_a^b f = \lim_{k\to\infty} \int_a^b \phi_k = 0.$$

5. (Q.5) Let $f(x_0\pm) = \lim_{y\to x_0\pm} f(y)$. According to the Theorem cited in the hint, if $f\in R[a,b]$, then $f(x\pm)$ exists for any $x\in(a,b)$ and the corresponding one-sided limits exist at the boundary points.

   The first step is to show that the set of points

$$S_\epsilon = \{x\in[a,b] : |f(x+)-f(x-)| > \epsilon\}$$

   is **finite**. Notice that S_ϵ is the set of all points of discontinuity of f with the jump size greater than ϵ. As f is regulated, there exists $\phi\in S[a,b]$ such that for any $x\in[a,b]$,

$$|f(x)-\phi(x)| < \epsilon/4.$$

   Taking one-sided limits of the above double inequality, we find after a simple manipulation:

$$f(x+)-f(x-)-\epsilon/2 \le \phi(x+)-\phi(x-) \le f(x+)-f(x-)+\epsilon/2.$$

   Take $x\in S_\epsilon$. As $|f(x+)-f(x-)|>\epsilon$, the above inequality implies that $\phi(x+)-\phi(x-)\neq 0$. But the set of discontinuities of a step function is finite. As we have just showed, that S_ϵ is contained in the set of discontinuities of ϕ, we can conclude that $|S_\epsilon|<\infty$.

   Finally, let us consider the sequence of sets $S_{1/n}$, $n\in N$. As it is easy to check, any point of discontinuity of f is in $S_{1/n}$ for some n. Note that

$$S_1 \subset S_{1/2} \subset S_{1/3} \subset \ldots.$$

   The enumeration of the set of the set of all discontinuities can be done inductively: the points of S_1 can be enumerated as S_1 is finite. Assume that all points in $S_{1/n}$ have been enumerated by $1,2,\ldots N_n$. As the set $S_{1/(n+1)}\setminus S_{1/n}$ is finite, we can enumerate its points starting with N_n+1. The result is the enumeration of all points in $S_{1/(n+1)}$. As any discontinuity of f is contained in $S_{1/n}$ for some n, we have established a one-to-one correspondence between the set of all discontinuities of f and $\mathbb{N}$, which means that this set is countable by definition.

0.2 Integration.

6. (Q.6) Let $P_N = \frac{5}{N}\{0,1,2,\ldots N\}$ be a sequence of partitions of $[0,5]$. Let ϕ_N : $\phi_N(0)=0, \phi_N\mid_{(5/N)(k-1,k]} = ((5/N)\cdot k)^2, k=1,2,\ldots,N$ be a sequence of step functions. Then $||f-\phi_N||_\infty \le \max_{1\le k\le N}(((5/N)\cdot k)^2 - ((2/N)\cdot(k-1)^2)) \le 50/N \to 0$ as $N\to\infty$. Therefore, ϕ_N's converge to f uniformly. By definition,

$$\int_0^5 f = \lim_{N\to\infty}\int_0^5 \phi_N = \lim_{N\to\infty}\frac{5}{N}\sum_{k=1}^N (5/N\cdot k)^2 == \lim_{N\to\infty}\left(\frac{5}{N}\right)^3\sum_{k=1}^N k^2$$

   But $\sum_{k=1}^N k^2 = \partial_x^2\sum_{k=1}^N e^{kx}\mid_{x=0} = \partial_x^2\frac{e^{(N+1)x}-1}{e^x-1}\mid_{x=0} = (1/3)N^3+(1/2)N^2+(1/6)N$. Substituting this into the previous formula and computing the limit, we get

$$\int_0^5 f = \frac{5^3}{3}.$$

7. (Q.7) By additivity, $I = -\int_2^a \frac{1}{\log(x)}dx + \int_2^b \frac{1}{\log(x)}dx$. Each of the terms is differentiable by FTC2. As $\partial_a F(b) = 0$, $\partial_a I = -1/\log(a)$, $\partial_b I = 1/\log(b)$.

8. (Q.8) $I(x) = -J(a(x)) + J(b(x))$, where $J(a) = \int_2^a 1/\log(t)dt$. As J is differentiable on the domain of a, b and a, b are differentiable on $\mathbb{R}$, the function I is differentiable on $\mathbb{R}$. So, we can apply the chain rule:

$$I'(x) = -J'(a(x))a'(x) + J'(b(x))b'(x) = -\frac{a'(x)}{\log a(x)} + \frac{b'(x)}{\log b(x)}.$$

9. (Q.9)
   (a) $F'(x) = \log^{10}(e^x)e^x = x^{10}e^x$
   (b) $G'(x) = 2x\sqrt{1+(1+x^2)^4} - 2x\sqrt{1+x^8}$
   (c) $H'(x) = 2x(e^{-x^{10}} + e^{-x^{10}}) = 4xe^{-x^{10}}$
   (d) $I'(x) = \frac{1}{2\sqrt{x}}\cos(x^2) + \frac{1}{x^2}\cos(1/x^4)$

10. (Q.10) For all the integrals below the integrands are continuous on the respective intervals of integration and we can apply the fundamental theorem of calculus for their evaluation.
   (a) Substitution: $t = \log(x)$. Then $\int_1^e \sin(\log(x))/x dx = \int_0^1 \sin(t)dt = 1 - \cos(1)$.
   (b) $\int_{\log(2)}^{\log(3)} 1/\cosh^2(x)dx = \int_{\log(2)}^{\log(3)} \left(\frac{sinh(x)}{cosh(x)}\right)' dx = \tanh(\log(3)) - \tanh(\log(2)) = 1/5$.
   (c) Substitution: $x = \sin(t)$. Then $\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}dx = \int_0^{\pi/4} \text{sign}(\cos(t))dt = \pi/4$.
   (d) Substitution: $t = \log(x)$. Then $\int_e^{e^2} \frac{1}{x\log(x)}dx = \int_1^2 \frac{1}{t}dt = \log(2)$.
   (e) $\int_{-1}^1 x^{1001}e^{-x^{100}}dx = 0$ as the integrand is odd and the limits of integration are symmetric.

0.3 Improper integrals.

11. (Q.11) Let $b_n = \int_{n-1}^n F$, $a_n = \int_{n-1}^n f$. Note that $|a_n| \leq \int_{n-1}^n |f| \leq b_n$. The fact that $\int_0^\infty F$ converges implies that the series

$$\sum_{n=1}^\infty b_n < \infty$$

As $|a_n| < b_n$, the series $\sum_{n=1}^\infty a_n$ converges absolutely by the comparison test for series (Analysis I). Therefore, $\lim_{N\to\infty} \int_0^N f$ exists and is finite. To finish the proof, we need to show that $\int_0^{R_n} f$ converges for any choice of sequence $R_n : \lim_{n\to\infty} R_n = \infty$. Let $\lceil x \rceil$ be the smallest integer greater or equal than x. Then

$$|\int_0^R f - \int_0^{\lceil R \rceil} f| \leq \int_R^{\lceil R \rceil} |f| \leq \int_{\lceil R \rceil - 1}^{\lceil R \rceil} |f| \leq b_{\lceil R \rceil} \overset{R\to\infty}{\longrightarrow} 0,$$

as $\sum_{n=1}^{\infty} b_n$ converges. Therefore,

$$\lim_{R\to\infty}\int_0^R f = \lim_{R\to\infty}\int_0^{\lceil R\rceil} f,$$

where the right hand side has already been shown to converge.

12. (Q.12) The integrals $\int_0^\infty e^{-x^3}$ and $\int_1^\infty e^{-x^3}dx$ diverge and converge simultaneously. But $e^{-x^3} \le e^{-x}$ for any $x \ge 1$. Therefore,

$$\int_1^\infty e^{-x^3} \le \int_1^\infty e^{-x}dx = 1.$$

We conclude that $\int_0^\infty e^{-x^3}dx$ converges by the comparison principle of Question 11.

13. (Q.13) $(1-x^4) = (1-x^2)(1+x^2) = (1-x)(1+x)(1+x^2) \ge (1-x)$ for $x \in [0,1)$. Therefore,

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}dx \le \int_0^1 \frac{1}{\sqrt{1-x}} < \infty.$$

Therefore the elliptic integral converges by the comparison principle.

14. (Q.14) $\int_{-R}^R x^3dx = \frac{x^4}{4}\mid_{-R}^R = 0$. However, $\int_{-\infty}^\infty xdx$ is divergent, as

$$\lim_{R_1\to\infty}\int_{-R_1}^0 x^3dx + \lim_{R_2\to\infty}\int_0^{R_2} x^3dx$$

does not exist.

15. (Q.15)

(a) $\int_0^{100} \frac{1}{x^{1/3}+2x^{1/4}+x^3}dx \le \int_0^{100} \frac{1}{x^{1/3}}dx = \lim_{\epsilon\downarrow 0}\int_\epsilon^{100} \frac{1}{x^{1/3}}dx = \lim_{\epsilon\downarrow 0} \frac{3}{2}x^{2/3}|_\epsilon^{100} < \infty$. So the integral converges by the comparison principle.

(b) The integral $\int_1^\infty \frac{\sin(x)}{x^2}$ converges. (Use the comparison principle with $F(x) = 1/x^2$.) Therefore, $\int_0^1 \frac{\sin(x)}{x^2}$ and $\int_0^\infty \frac{\sin(x)}{x^2}dx$ diverge and converge simultaneously. But

$$\lim_{\epsilon\downarrow 0}\int_\epsilon^1 \frac{\sin(x)}{x^2} = -\lim_{\epsilon\downarrow 0}\left(\frac{\sin(x)}{x}\mid_\epsilon^1 - \int_\epsilon^1 \frac{\cos(x)}{x}dx\right)$$

$$= -\lim_{\epsilon\downarrow 0}\left(\frac{\sin(x)}{x}\mid_\epsilon^1 - \cos(x)\log(x)\mid_\epsilon^1 - \int_\epsilon^1 \sin(x)\log(x)dx\right),$$

which diverges logarithmically (verify that $\int_0^1 \log(x)\sin(x)dx$ converges!). Therefore, $\int_0^\infty \frac{\sin(x)}{x^2}dx$ diverges.

0.4 Uniform convergence.

16. (Q.16) For any $x \in (-1,1)$, $\lim_{n\to\infty}\sum_{k=0}^{n-1} x^k = \frac{1}{1-x} := f(x)$ pointwise. Let $f_n(x) = \sum_{k=1}^{n-1} x^k = \frac{1-x^n}{1-x}$. Choose $x_n = 1-1/n \in (-1,1)$. Then

$$\lim_{n\to\infty}|f_n(x_n) - \frac{1}{1-x_n}| = \lim_{n\to\infty}|\frac{x_n^n}{1-x_n}| = \lim_{n\to\infty} n(1-1/n)^n = +\infty.$$

So choosing $\epsilon = 1$ in the negation of the definition of the uniform convergence, we can always find n and $x \in (-1, 1)$ such that $|f(x) - f_n(x)| > 1$. Therefore, $\sum_k x^k$ does not converge uniformly in $(-1, 1)$. On the other hand, for any $x \in [-R, R]$, where $0 \leq R < 1$,

$$|f_n(x) - f(x)| \leq \frac{R^n}{1-R} \to 0 \text{ for } n \to 0,$$

which immediately implies the uniform convergence on $[-R, R]$.

17. (Q.17) Without restricting generality, we can consider symmetric intervals $[-R, R]$. For any $x \in [-R, R]$,

$$\begin{aligned}
|\sum_{k=1}^{n-1} \frac{x^k}{k!} - e^x| &\leq \sum_{k=n}^{\infty} \frac{R^k}{k!} = \frac{R^k}{k!} \sum_{k=n}^{\infty} \frac{R^{k-n}}{(n+1)(n+2)\ldots(n+(k-n))} \\
&\leq \frac{R^k}{k!} \sum_{k=0}^{\infty} \frac{R^k}{k!} = \frac{R^k}{k!} e^R \to 0 \text{ as } k \to \infty,
\end{aligned}$$

where the last step can be justified using Stirling's formula. Therefore, the exponential sequence converges uniformly on $[-R, R]$, hence on any finite subinterval of $\mathbb{R}$.

18. (Q.18) $\Rightarrow$. If $f_n \to f$ uniformly on A, then for any $\epsilon > 0$ there is $N \in \mathbb{N}$: for any $n > N$ and any $x \in A$, $|f(x) - f_n(x)| < \epsilon/2$. Therefore, $\sup_{x \in A} |f_n(x) - f(x)| < \epsilon$ and we get the definition of $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$.

   $\Leftarrow$. If $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon$, then for any $x \in A$ $|f_n(x) - f(x)| < \epsilon$ and the the uniform convergence follows immediately from $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$.

19. (Q.19) Let $S_n = x^2 - x^4 + x^4 - x^6 + \ldots + x^{2n} - x^{2n+2} = x^2 - x^{2n+2}$. Clearly, $\lim_{n\to\infty} S_n(x) = x^2$ for $|x| < 1$ and $\lim_{n\to\infty} S_n(x) = 0$ for $|x| = 1$. Consider $x_n = 1 - 1/n \in (-1, 1)$. Then for any $\delta > 0$, there is $n_\delta \in \mathbb{N}$ such that for any $n > n_\delta$

$$x_n^2 - S_n = x_n^{2n+2} = (1 - 1/n)^{2n+2} \geq (1 - 1/n)^n = e^{n \log(1-1/n)} \geq e^{-1-\delta}..$$

   Therefore, $\lim_{n\to\infty} \sup_{x\in(-1,1)} |S_n(x) - x^2| \neq 0$ and the convergence cannot be uniform by Question 18.

20. (Q.20) Suppose the convergence $F_n \to F$ were not uniform. Negating the definition of the uniform convergence, we conclude that there must exist $c > 0$ and a strictly increasing sequence $\{p_n\}_{n\geq 1} \subset \mathbb{N}$ such that

$$\sup_{x\in[a,b]} (F_{p_n}(x) - F(x)) > c, \ n = 1, 2, \ldots$$

   As $F_{p_n} - F$ is continuous on the closed interval $[a, b]$, it reaches its maximal value at some $x_{p_n} \in [a, b]$. The sequence $\{x_{p_n}\}_{n\geq 1}$ is bounded. By the Boltzano-Weierstrass theorem it contains a subsequence $\{x_{p_{n_k}}\}_{k\geq 1}$, which converges to a point $x_0 \in [a, b]$. By construction,

$$F_{p_{n_k}}(x_{p_{n_k}}) - F(x_{p_{n_k}}) > c, \ k = 1, 2, \ldots$$

As $F_n(x) > F_m(x)$ for any x and $m > n$, we conclude that

$$F_{p_n}(x_{p_{n_k}}) - F(x_{p_{n_k}}) > c.$$

for each $n < n_k$. Taking the limit $k \to \infty$ in the above inequality and using the continuity, we find that

$$F_{p_n}(x_0) - F_p(x_0) \geq c > 0, \; n = 1, 2, 3, \ldots,$$

which contradicts the pointwise convergence $F_n \to F$ at x_0. Therefore, F_n must converge to F uniformly.

November the 9th, 2015 Sergey Nazarenko and Oleg Zaboronski.