

15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to **three** randomly chosen 'B' and **one** 'A' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' questions.

The answers to ALL questions are to be submitted by the deadline of 3pm on Monday, the 17th of November 2014. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeon loft if you are a Maths student, or in the drop box labeled with your teaching assistant's name, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

0.1 Further exercises in integration

1. A. Test the convergence of the probability integral

$$\int_0^{\infty} e^{-x^2} dx$$

Hint. Use the following comparison principle: if $\int_a^{\infty} F$ converges and $|f| < F$, then $\int_a^{\infty} f$ converges.

2. B. Prove that the Euler integrals of the first kind (beta-function) converges when $p > 0$ and $q > 0$:

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx.$$

3. B. Prove that the Euler integral of the second kind (gamma-function) converges for $p > 0$:

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx.$$

4. B. Prove the following generalization of the fundamental theorem of calculus: Let $F : [a, b] \rightarrow \mathbb{R}$ be continuous and piecewise continuously differentiable. Then

$$\int_a^b F'(x) dx = F(b) - F(a).$$

Remark. The definition of $\int_a^b F'$: F' is not defined at a finite set of points of $[a, b]$ where it has jump discontinuities. However, F' has one sided-limits at every point of $[a, b]$. Let us assign any values we like to F' at the points of discontinuity. We get a regulated function on $[a, b]$ which can be integrated. The value of the integral is denoted by $\int_a^b F'$. Notice that arbitrary choices we made to extend F' to the whole of $[a, b]$ do not influence the value of the integral.

5. A. Use the theorem of question 4 to calculate

$$\int_0^{\pi n} \text{sign}(\sin(x)) dx,$$

for $n = 1, 2, 3, \dots$. Here $\text{sign} : \mathbb{R} \rightarrow \mathbb{R}$ is a sign function: $\text{sign}(x) = 1$ for $x > 0$, $\text{sign}(x) = -1$, $\text{sign}(0) = 0$.

Hint. Sketch the graphs of $\sin(x)$, $\text{sign}(\sin(x))$ - this will help you to guess the shape of F : $F'(x) = \text{sign}(\sin(x))$ for $x \neq \pi k$, $k \in \mathbb{Z}$.

6. A. Let $f : [-a, a] \rightarrow \mathbb{R}$ be continuous, $a > 0$. Prove or disprove the following two statements:

(a) If f is an even function ($f(x) = f(-x)$),

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx.$$

(b) If f is an odd function ($f(x) = -f(-x)$),

$$\int_{-a}^a f(x)dx = 0.$$

7. B. Let n be a natural number. Use integration by parts to show that for the integral

$$I_n = \int_0^{\pi/2} \cos^n(x)dx$$

the following reduction formula holds true:

$$I_n = \frac{n-1}{n} I_{n-2}, n > 1.$$

Use the reduction formula to find I_n , then calculate I_9, I_{10} .

8. B. Use definite integrals to find the limits of the following sums:

(a) $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n-1}{n^2} \right)$

(b) $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} \right)$

(c) $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}}, p > 0$

9. A. Determine the sign of the following integral:

$$\int_0^{2\pi} f(x)dx,$$

where $f(0) = 1$, $f(x) = \frac{\sin(x)}{x}$ for $x > 0$.

10. A. Determine (without evaluating) which of the following integrals is greater:

- $\int_0^1 \sqrt{1+x^2}dx$ or $\int_0^1 1dx$
- $\int_0^1 x^2 \sin^2(x)dx$ or $\int_0^1 x \sin^2(x)dx$
- $\int_1^2 e^{x^2} dx$ or $\int_1^2 e^x dx$

0.2 Uniform convergence.

11. B. Find $f'(x)$ if

$$f(x) = \int_x^{10} e^{-xy^2} dy, 0 \leq x \leq 10.$$

Justify all steps of the calculation.

Hint. Let $I(a, b, c) = \int_a^b dx f(x, c)$. You need to show that f satisfies all conditions for

necessary for FTC1 and the theorem on the differentiation of integrals depending on a parameter to hold. Then

$$I'(a(x), b(x), c(x)) = \frac{\partial I}{\partial a}(a(x), b(x), c(x))a'(x) + \frac{\partial I}{\partial b}(a(x), b(x), c(x))b'(x) + \frac{\partial I}{\partial c}(a(x), b(x), c(x))c'(x),$$

and each of the partial derivatives can be evaluated using either FTC1 or the theorem on the differentiation of integrals depending on a parameter.

12. B. Applying differentiation with respect to a parameter α , evaluate the following integral

$$\int_0^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx, \quad \alpha, \beta > 0.$$

Hint. You may exchange the order of integration and differentiation without a justification. The exchange is justified due to the uniform convergence of the above improper integral depending on parameters. The theory of improper integrals depending on parameters is not currently covered by our course.

13. B. The Laplace transform of $f : [0, \infty) \rightarrow \mathbb{R}$ is a function $F : (0, \infty) \rightarrow \mathbb{R}$ defined by the formula

$$F(p) = \int_0^{\infty} e^{-pt} f(t) dt, \quad p > 0.$$

Find the Laplace transform of the following functions:

- (a) $f(t) = 1$
- (b) $f(t) = e^{-\alpha t}$, $\alpha > 0$
- (c) $f(t) = \cos(\beta t)$, $\beta \in \mathbb{R}$

Notice that the Laplace transform of a bounded function is not necessarily bounded.

14. B. Prove the uniform convergence of the functional series in the indicated intervals:

- (a) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $x \in [-1, 1]$
- (b) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$, $x \in \mathbb{R}$

15. B. Applying term-wise differentiation and integration, find the sums of the series in the indicated intervals:

- (a) $\sum_{k=1}^{\infty} \frac{x^k}{k}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$
- (b) $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$
- (c) $\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$.

Justify all steps of your calculations.