Questions for credit: 6 (4 points), 8 (5 points), 12 (8 points) and 15 (8 points)

0.1 Further exercises in integration

1. $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$. Therefore, $\int_0^\infty e^{-x^2}dx$ and $\int_1^\infty e^{-x^2}dx$ are either both convergent or both divergent. The exponential function is increasing and $-x^2 \leq -x$ for $x \geq 1$. Therefore, $e^{-x^2} \leq e^{-x}$ for $x \geq 1$. But $\int_1^\infty e^{-x} := \lim_{R\to\infty}\int_1^R e^{-x}dx = \lim_{R\to\infty}(e^{-1} - e^{-R}) = e^{-1}$, convergent. Therefore, the probability integral converges by the comparison principle.

2. $B(p,q) = \int_0^{1/2} x^{p-1}(1-x)^{q-1}dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}dx$. The Euler integral converges iff each of the integrals in the right hand side converges. For $x \in (0, 1/2]$, $x^{p-1}(1-x)^{q-1} \leq x^{p-1}max(1,(1/2)^{q-1})$. The integral $\int_0^{1/2} x^{p-1}dx$ converges for $p > 0$ (lecture material). Therefore, $\int_0^{1/2} x^{p-1}(1-x)^{q-1}dx$ converges by the comparison principle. Similarly, for $x \in [1/2, 1)$, $x^{p-1}(1-x)^{q-1} \leq (1-x)^{q-1}max(1,(1/2)^{p-1})$. The integral $\int_{1/2}^1 (1-x)^{q-1}dx = \int_0^{1/2} x^{q-1}dx$ converges for $q > 0$ (lecture material). Therefore, $\int_{1/2}^1 x^{p-1}(1-x)^{q-1}dx$ converges by the comparison principle. We conclude that the Euler integral of the first kind converges for $p > 0, q > 0$.

3. $\int_0^\infty x^{p-1}e^{-x}dx = \int_0^1 x^{p-1}e^{-x}dx + \int_1^\infty x^{p-1}e^{-x}dx$. The Euler integral converges iff each integral in the right hand side converges. For $x \in (0,1]$, $x^{p-1}e^{-x} \leq x^{p-1}$. The integral $\int_0^1 x^{p-1}dx$ converges for $p > 1$ (lecture material). Therefore, $\int_0^1 x^{p-1}e^{-x}dx$ converges by the comparison principle. For $x \in [1,\infty)$, $x^{p-1}e^{-x} \leq x^{|p-1|}e^{-x} = (x^{|p-1|}e^{-x/2})e^{-x/2} \leq \sup_{x\in[0,\infty)}(x^{|p-1|}e^{-x/2})e^{-x/2} \leq C_p e^{-x/2}$, where $C_p > 0$ is a p-dependent constant. (The global maximum of $g(x) = x^{|p-1|}e^{-x/2}$ is achieved at $x_c = 2|p-1|$. The value at the global maximum is $C_p = (2|p-1|)^{|p-1|}e^{-|p-1|} > 0$.) The integral $\int_1^\infty e^{-x/2}dx$ converges (question 1). Therefore, $\int_1^\infty x^{p-1}e^{-x}dx$ converges by the comparison principle. We conclude that the Euler integral of the second kind converges for $p > 0$.

4. Let $\{p_0 = a < p_1 < \ldots < p_N = b\}$ be a finite set of points such that F is continuously differentiable on each of $[p_{k-1}, p_k]$, $k = 1, 2, \ldots N$. Such a set exists by the definition of piece-wise continuous differentiability. By FTC2, $\int_{p_{k-1}}^{p_k} F' = F(p_k) - F(p_{k-1})$. Therefore,

$$\int_a^b F' = \sum_{k=1}^N \int_{p_{k-1}}^{p_k} F' = \sum_{k=1}^N (F(p_k) - F(p_{k-1})) = F(b) - F(a).$$

   Notice the crucial role the continuity of F plays in the proof.

5. Let $k \in \mathbb{Z}$. $sign(sin(x)) = 1$ for $x \in (2k\pi, (2k+1)\pi)$ and $sign(sin(x)) = -1$ for $x \in ((2k+1)\pi, (2k+2)\pi)$. Let F: $F(x) = x - 2k\pi$ for $x \in [2k\pi, (2k+1)\pi]$, $F(x) = \pi - (x - (2k+1)\pi)$ for $x \in [(2k+1)\pi, (2k+2)\pi]$. (A 'saw tooth' function.) Notice that F is continuous, piece-wise continuously differentiable and $F'(x) = sign(sin(x))$ for $x \neq \pi k$. By the theorem of Question 4, $\int_0^{\pi n} sign(sin(x))dx = \int_0^{\pi n} F' = F(\pi n) - F(0) = F(\pi n)$, which is zero for even n and π for odd n.

6. In each case $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \overset{t=-x}{=} -\int_a^0 f(-t)dt + \int_0^a f(x)dx = \int_0^a (f(x) + f(-x))dx$. Therefore:

   (a) If $f(x) = f(-x)$, $\int_{-a}^{a} f = 2\int_0^a f$.

   (b) If $f(x) = -f(-x)$, $\int_{-a}^{a} f = \int_0^a 0 \cdot dx = 0$.

   Both statements are true.

7. $I_n = \int_0^{\pi/2} \sin'(x)\cos^{n-1}(x)dx = \sin(x)\cos^{n-1}(x)|_{x=0}^{\pi/2} - \int_0^{\pi/2} \sin(x)(cos^{n-1})'(x)dx = (n-1)\int_0^{\pi/2} \sin^2(x)\cos^{n-2}(x)dx = (n-1)\int_0^{\pi/2}(1-\cos^2(x))\cos^{n-2}(x)dx = (n-1)(I_{n-2} - I_n)$. So,

$$I_n = \frac{n-1}{n} I_{n-2}.$$

   By explicit calculation, $I_0 = \pi/2$, $I_1 = 1$.

   If n is even,

$$\begin{aligned} I_n &= \frac{n-1}{n} I_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} I_{n-4} = \ldots = \frac{(n-1)(n-3)\ldots 3\cdot 1}{n(n-2)\ldots 4\cdot 2} I_0 \\ &= \frac{(n-1)(n-3)\ldots 3\cdot 1}{n(n-2)\ldots 4\cdot 2}\frac{\pi}{2}. \end{aligned}$$

   If n is odd,

$$\begin{aligned} I_n &= \frac{n-1}{n} I_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} I_{n-4} = \ldots = \frac{(n-1)(n-3)\ldots 2\cdot 4}{n(n-2)\ldots 3\cdot 1} I_1 \\ &= \frac{(n-1)(n-3)\ldots 4\cdot 2}{n(n-2)\ldots 3\cdot 1} \end{aligned}$$

   So,

$$I_{10} = \frac{9\cdot 7\cdot 5\cdot 3}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{63\pi}{2^9} = \frac{63\pi}{512},$$

$$I_9 = \frac{8\cdot 6\cdot 4\cdot 2}{9\cdot 7\cdot 5\cdot 3} = \frac{2^7}{315} = \frac{128}{315}.$$

8. The theoretical underpinning for this question is Theorem 22 of the course. Let $x_k = k/n$, $k = 1, \ldots, n-1$.

   (a)

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{k}{n^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} x_k \overset{Thm.22}{=} \int_0^1 x dx = \frac{1}{2}.$$

   (b)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+x_k} \overset{Thm.22}{=} \int_0^1 \frac{1}{1+x}dx = \log(1+x)|_0^1 = \log(2).$$

   (c)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^p}{n^{p+1}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} x_k^p \overset{Thm.22}{=} \int_0^1 x^p dx = \frac{1}{p+1}.$$

9. The integral is positive: f is continuous on $[0, 2\pi]$, so the integral exists in the usual sense.

$$\int_0^{2\pi} f = \int_0^{\pi} f + \int_{\pi}^{2\pi} f = \int_0^{\pi} (f(x) + f(x+\pi))dx,$$

$f(x) + f(x+\pi) = sin(x)/x + sin(x+\pi)/(x+\pi) = sin(x)(1/x - 1/(x+\pi)) = sin(x)\frac{\pi}{x(x+\pi)} > 0$ for $x \in (0, \pi)$. Therefore, $\int f > 0$ by integral bounds.

10. (a) $\int_0^1 \sqrt{1+x^2}dx - \int_0^1 1 \cdot dx = \int_0^1 (\sqrt{1+x^2} - \sqrt{1+0})dx > 0$, as the integrand is continuous, non-negative, and positive at at least one point of the integration interval. So, $\int_0^1 \sqrt{1+x^2}dx > \int_0^1 1 \cdot dx$

    (b) $\int_0^1 x^2 \sin^2(x)dx - \int_0^1 x \sin^2(x)dx = \int_0^1 x(x-1)\sin^2(x)dx < 0$, as the integrand is continuous, non-positive, and negative at at least one point of the integration interval. So, $\int_0^1 x^2 \sin^2(x)dx < \int_0^1 x \sin^2(x)dx$

    (c) $\int_1^2 e^{x^2}dx - \int_1^2 e^x dx = \int_1^2 e^x(e^{x^2-x} - 1)dx > 0$, as the integrand is continuous, non-negative ($x^2 - x \geq 0$ for $x \geq 1$, so $e^{x^2-x} \geq 1$), and positive at at least one point of the integration interval. So, $\int_1^2 e^{x^2}dx > \int_1^2 e^x dx$

0.2 Uniform convergence.

11. The integrand is continuous (hence uniformly continuous) on $D = [0, 10] \times [0, 10]$. More over, the partial derivative of the integrand with respect to x is also continuous on D. Therefore, the integral is differentiable with respect to the limits of integration (FTC1) and the c-derivative of the integral $\int_a^b e^{-cy^2}dy$ is equal to the integral of the c-derivative of the integrand. Therefore, applying the chain rule formula from the hint we find

$$f'(x) = -e^{-x^3} - \int_x^{10} y^2 e^{-xy^2}dy$$

12. Let $F(\alpha) = \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}dx$. Then $F'(\alpha) = \int_0^\infty -e^{-\alpha x} = -\frac{1}{\alpha}$. Therefore, $F(\alpha) = -\log(\alpha) + C(\beta)$, where $C(\beta$ is a constant of integration. It can be found by noticing that $F(\beta) = 0$, which gives $C(\beta) = \log(\beta)$. Therefore, $F(\alpha) = \log(\beta/\alpha)$.

13. (a) $F(p) = \int_0^\infty e^{-pt}dt = \frac{1}{p}$

    (b) $F(p) = \int_0^\infty e^{-pt}e^{-\alpha t}dt = \frac{1}{p+\alpha}$

    (c) $F(p) = \int_0^\infty e^{-pt}cos(\beta t)dt = \frac{1}{2i}(\int_0^\infty e^{-pt}e^{i\beta t}dt + \int_0^\infty e^{-pt}e^{-i\beta t}dt) = \frac{1}{2}(\frac{1}{p-i\beta} + \frac{1}{p+i\beta}) = \frac{p}{p^2+\beta^2}$

14. (a) For any $x \in [-1, 1]$, $|x^n/n^2| \leq 1/n^2$. As $\sum_{n=1}^\infty \frac{1}{n^2} < \infty$, $\sum_{n=1}^\infty \frac{x^n}{n^2}$ converges uniformly on $[-1, 1]$ by the M-test.

    (b) For any $x \in \mathbb{R}$, $sin(nx)/2^n \leq 1/2^n$. But $\sum_{n=1}^\infty \frac{1}{2^n} < \infty$. So, $\sum_{n=1}^\infty \frac{\sin(nx)}{2^n}$ converges on $\mathbb{R}$ uniformly by the M-test.

15. In each of the three cases below, both the power series $S(x)$ and the series for $S'(x)$ converge uniformly on $[a, b]$. Therefore, the term-wise differentiation is justified. Integrating S' to obtain S is done using $FTC2$.

(a) Let $S(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. Then $S'(x) = \sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$. Therefore, $S(x) = -log(1-x) + C$, where C is an integration constant. As $S(0) = 0$, $C = 0$. Therefore, $\sum_{k=1}^{\infty} \frac{x^k}{k} = -log(1-x)$.

(b) Let $S(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$. Then $S'(x) = \sum_{k=1}^{\infty} (-x)^{k-1} = \frac{1}{1+x}$. Therefore, $S(x) = \log(1+x) + C$, where C is an integration constant. As $S(0) = 0$, $C = 0$. Therefore, $\sum_{k=1}^{\infty} \frac{x^k}{k} = log(1+x)$.

(c) Let $S(x) = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}$. Then $S'(x) = \sum_{k=1}^{\infty} x^{2k-2} = \frac{1}{1-x^2}$. Therefore, $S(x) = \frac{1}{2} \log \frac{1+x}{1-x} + C$, where C is an integration constant. As $S(0) = 0$, $C = 0$. Therefore, $\sum_{k=1}^{\infty} \frac{x^k}{k} = -\frac{1}{2} \log \frac{1+x}{1-x}$.

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