

15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to **three** randomly chosen 'B' and **one** 'A' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' questions.

The answers to ALL questions are to be submitted by the deadline of 3pm on Monday, the 23rd of November 2015. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeonloft if you are a Maths student, or in the dropbox labelled with the course's code, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

0.1 Further exercises in integration

1. A. Prove that the Euler integrals of the first kind (beta-function) converges when $p > 0$ and $q > 0$:

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx.$$

Hint. Use the comparison test which you studied in Question 11, Assignment 2. The same test will be useful for exercises 4 and 7 below.

2. B. Using repeated integration by parts evaluate the beta-function defined in Question 1 for the particular case when p and q are positive integers
3. B. Express the following integral in terms of the beta function defined in Question 1:

$$I_{n,m} = \int_0^{\pi/2} \sin^m(x) \cos^n(x) dx,$$

where m, n are positive integers.

4. A. Prove that the Euler integral of the second kind (gamma-function) converges for $p > 0$:

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx.$$

5. B. Show that for the gamma-function defined in Question 4 the following reduction formula holds true:

$$\Gamma(p+1) = p\Gamma(p), \quad p > 0.$$

From this derive that $\Gamma(n+1) = n!$, $n = 1, 2, 3, \dots$

6. B. Derive formally the following relation between the Euler integrals defined in Questions 1 and 4:

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p > 0, q > 0.$$

Hint. Write $\Gamma(p)\Gamma(q)$ as a repeated integral and use the following changes of integration variables: firstly, change from (u, v) to (x^2, y^2) in the product of gamma-integrals; secondly, re-write the repeated integral as a double integral (a formal step); thirdly, change to polar coordinates $x = r \cos(\phi)$, $y = r \sin(\phi)$ in the resulting double integral. You will end up with

a product of two one-dimensional integrals which can be computed using the formula you derived in Question 3: as it turns out, it is valid for any $m, n > -1$, not just for natural m, n . Assume this generalisation without a proof.

7. B. Let K_α be the modified Bessel function of the second kind of order $\alpha \geq 0$:

$$x \mapsto K_\alpha(x) = \int_0^\infty e^{-x \cosh(t)} \cosh(\alpha t) dt, \quad x \in (0, \infty).$$

Fix $\alpha > 0$ and analyse the convergence of the integral defining K_α for $x \in [0, \infty)$.

8. B. Use the Fundamental Theorem of Calculus to prove the Leibnitz Integral Rule: let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous; let $u, d : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable such that $u(x) \geq d(x)$ for any $x \in \mathbb{R}$. Then

$$\frac{d}{dx} \int_{d(x)}^{u(x)} f(t) dt = f(u(x)) \cdot u'(x) - f(d(x)) \cdot d'(x).$$

9. A. Let $f \in C[-1, 1]$. Show that

$$\int_0^{\pi/2} f(\cos(x)) dx = \int_0^{\pi/2} f(\sin(x)) dx$$

10. B. Using integration by parts, prove that

$$0 < \int_{100\pi}^{200\pi} \frac{\cos(x)}{x} < \frac{1}{100\pi}$$

0.2 Uniform convergence.

11. A. Let $A \subset \mathbb{R}$. Prove that a uniformly continuous function on A is also continuous at every point of A .
12. A. Show that $f(x) = x^2$ on \mathbb{R} is continuous but not uniformly continuous. Find an example of a uniformly continuous function on \mathbb{R} which is bounded but not constant.
13. B. Prove the following extremely useful theorem: Let f_n, f, g be regulated functions on $[a, \infty)$ such that (i) for every $b > a$, $f_n \rightarrow f$ uniformly on $[a, b]$, (ii) for all $x \geq a$, $|f_n(x)| \leq g(x)$, and (iii) $\int_a^\infty g$ is finite. Then the improper integral $\int_a^\infty f$ exists and $\int_a^\infty f_n \rightarrow \int_a^\infty f$ as $n \rightarrow \infty$.

Remarks. The function g is said to dominate the sequence (f_n) , which gives the theorem its name - the dominated convergence theorem. Amazingly, the statement of the theorem remains true if the condition of uniform convergence is replaced with mere pointwise convergence.

14. B. Prove that for any $x > 1$,

$$\lim_{n \rightarrow \infty} \int_0^n t^{x-1} (1 - t/n)^n dt = \Gamma(x)$$

Hint. Use the dominated convergence theorem, Question 13. The above result holds true for any $x > 0$, but the proof of the $x \leq 1$ case is a bit more involved due to a singularity at zero. If you find at some point that you need to estimate $\log(1 - t/n)$, we suggest that you use Taylor's theorem with the Lagrange remainder.

15. B. Prove that the zeta-function series

$$\zeta(x) := \sum_{k=1}^{\infty} \frac{1}{k^x}$$

converges uniformly in $[1 + \delta, \infty)$. Does it converge uniformly in $(1, \infty)$?

16. A. Prove the uniform convergence of the functional series in the indicated interval:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{\sqrt{n}}, \quad x \in [0, 1]$$

For the following 3 questions apply term-wise differentiation and integration to find the sums of the series in the indicated intervals. Justify all steps of your calculations.

17. A. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{2^{k-1}}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$

18. A. $\sum_{k=0}^{\infty} (k+1)x^k$, $x \in [a, b]$, $-1 < a < 0 < b < 1$

19. A. $\sum_{k=1}^{\infty} k(k+1)x^{k-1}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$.

20. B. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)3^{n-1}}$.

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Sergey Nazarenko and Oleg Zaboronski