NB. THESE ARE SKELETON SOLUTIONS ONLY, USE THEM WISELY!

0.1 Further exercises in integration

1. (Q.1) Let

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = L_1 + L_2,$$

where

$$L_1 = \lim_{\epsilon_1 \downarrow 0} \int_{\epsilon_1}^{1/2} x^{p-1}(1-x)^{q-1}dx,$$

$$L_2 = \lim_{\epsilon_2 \downarrow 0} \int_{1/2}^{\epsilon_2} x^{p-1}(1-x)^{q-1}dx.$$

The improper integral exists iff L_1, L_2 exists. Consider L_1. The function $(1-x)^{q-1}$ is bounded on $[0, 1/2]$. Let $B > 0$ be the bound. The improper integral

$$\lim_{\epsilon_1 \downarrow 0} \int_{\epsilon_1}^{1/2} Bx^{p-1}dx = \lim_{\epsilon_1 \downarrow 0} B\frac{x^p}{p} |_{\epsilon_1}^{1/2} = B\frac{(1/2)^p}{p},$$

exists (here we used that $p > 0$). Therefore, L_1 exists by the comparison test. Similarly, using the fact that x^p is bounded on $[1/2, 1]$ we can establish the existence of L_2. Therefore, the Euler integral of the first kind exists by definition.

2. (Q.2) If $p, q \in \mathbb{N}$, the Euler integral exists in the usual sense. The calculation goes as follows:

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_0^1 \left(\frac{x^p}{p}\right)'(1-x)^{q-1} = \frac{(q-1)}{p}\int_0^1 x^p(1-x)^{q-2}dx$$

$$= \frac{(q-1)(q-2)}{p(p+1)}\int_0^1 x^{p+1}(1-x)^{q-3}dx = \ldots = \frac{(q-1)(q-2)\ldots 1}{p(p+1)\ldots(p+q-2)}\int_0^1 x^{p+q-2}dx$$

$$= \frac{(q-1)!(p-1)!}{(p+q-1)!}.$$

3. (Q.3) $I_{n,m}$ exists in the usual sense. Consider the following change of variables: $y = \sin^2(x)$. Notice that the inverse function $x(y)$ is not differentiable at $y = 0, 1$. Therefore, the 'clean' computation goes as follows:

$$I_{n,m} = \lim_{\epsilon_1 \downarrow 0, \epsilon_2 \downarrow 0} \int_{\epsilon_1}^{1-\epsilon_2} \sin^m(x(y))\cos^n(x(y))x'(y)dy = \frac{1}{2}\int_0^1 y^{(m-1)/2}(1-y)^{(n-1)/2}dy$$

$$= \frac{1}{2}B((m+1)/2, (n+1)/2).$$

When resolving $y = sin^2(x)$ with respect to $\sin(x)$ we used $\sin(x) \geq 0, \cos(x) \geq 0$ for $x \in [0, \pi/2]$.

4. (Q.4) Let us represent $\Gamma(p) = L_1 + L_2$, where

$$L_1 = \lim_{\epsilon \downarrow 0} \int_\epsilon^1 x^{p-1} e^{-x} dx, L_2 = \lim_{R \uparrow 0} \int_1^R x^{p-1} e^{-x} dx.$$

L_1 converges by the comparison test which uses $x^{p-1}e^{-x} \leq x^{p-1}$ for $x \in (0,1]$ and the fact that $\int_0^1 x^{p-1}dx$ converges for any $p > 0$. L_2 also converges by the comparison test: notice that the function $x^{p-1}e^{-x/2}$ is bounded on $[1, \infty)$:

$$x^{p-1}e^{-x/2} \leq e^{-1/2} (p \leq 3/2), \;\; x^{p-1}e^{-x/2} \leq x_c^{p-1} e^{-x_c/2} (p \leq 3/2),$$

where $x_c = 2(p-1)$. Let us call this bound $B(p)$. We found that

$$x^{p-1}e^{-x} \leq B(p) e^{-x/2}, \;\; x \in [1, \infty).$$

As the integral $\int_0^\infty e^{-x/2}dx$ converges, L_2 exists by the comparison test. Therefore, the Euler integral of the second kind exists by definition.

5. (Q.5) For $p > 0$,

$$\Gamma(p) = \frac{1}{p} \int_0^\infty (x^p)' e^{-x} = \lim_{l \downarrow 0} \frac{x^p}{p} e^{-x}|_l^1 + \lim_{u \uparrow \infty} \frac{x^p}{p} e^{-x}|_1^u + \frac{1}{p}\Gamma(p+1) = \frac{1}{p}\Gamma(p+1)$$

The result $\Gamma(n+1) = n!$ follows from the above by induction started from $\Gamma(1) = 1$.

6. (Q.6)

$$\Gamma(p)\Gamma(q) = \int_0^\infty u^{p-1}e^{-u}du \int_0^\infty v^{q-1}e^{-v}dv = 4\int_0^\infty x^{2p-1}e^{-x^2} \int_0^\infty y^{2q-1}e^{-y^2}$$

$$= 4\int_0^\infty \int_0^\infty dxdy x^{2p-1} y^{2q-1} e^{-x^2-y^2} = 4\int_0^{\pi/2} \sin^{2p-1}(\phi)\cos^{2q-1}(\phi) \int_0^\infty r^{2(p+q)-1}e^{-r^2}$$

$$= B(p,q)\Gamma(p+q).$$

The second inequality in the above is the change of variables $(u,v) \to (x^2, y^2)$; the third inequality is a formal replacement of the repeated integral with a double integral; the fourth is a change to polar coordinates; the fifth equality uses an appropriate generalization of Question 3 and the change of variables $r^2 = x$.

7. (Q.7) If $x = 0$ and $\alpha > 0$, the Bessel integral diverges:

$$\int_0^\infty \cosh(\alpha t)dt := \lim_{R \uparrow \infty} \int_0^R \cosh(\alpha t)dt \geq \frac{1}{2} \lim_{R \uparrow \infty} \int_0^R e^{\alpha t}dt = \infty.$$

If $x > 0$ and $\alpha > 0$,

$$K_\alpha(x) \leq \int_0^\infty e^{-\frac{x}{2}e^t} e^{\alpha t}dt = \int_1^\infty e^{-\frac{x}{2}y} y^{\alpha-1}dy \leq \int_0^\infty e^{-\frac{x}{2}y}y^{\alpha-1}dy = (x/2)^{-\alpha}\Gamma(\alpha).$$

The right hand side of the above chain is finite as the gamma integral is finite for $\alpha > 0$, see Question 4. We conclude that $K_\alpha(x)$ converges for $x > 0$, $\alpha > 0$.

8. (Q.8) Let $F(x) = \int_0^x f$. As f is continuous, F is differentiable and $F' = f$ (FTC1). By FTC2,

$$\int_{d(x)}^{u(x)} f = F(u(x)) - F(d(x)).$$

Applying the chain rule,

$$\frac{d}{dx}\int_{d(x)}^{u(x)} f = F'(u(x))u'(x) - F'(d(x))d'(x) = f(u(x))u'(x) - f(d(x))d'(x).$$

9. (Q.9) Suggested change of variables: $y = \pi/2 - x$.

$$\int_0^{\pi/2} f(\cos(x))dx = -\int_{\pi/2}^{0} f(\cos(\pi/2 - y))dy = \int_0^{\pi/2} f(\sin(y))dy,$$

where we used $\cos(\pi/2 - y) = \sin(y)$

10. (Q.10) (i) Upper bound:

$$\int_{100\pi}^{200\pi} \frac{\cos(x)}{x}dx = \int_{100\pi}^{200\pi} \frac{\sin'(x)}{x}dx = \frac{\sin(x)}{x}\Big|_{100\pi}^{200\pi} + \int_{100\pi}^{200\pi} \frac{\sin(x)}{x^2}dx$$

$$= \int_{100\pi}^{200\pi} \frac{\sin(x)}{x^2}dx \le \int_{100\pi}^{200\pi} \frac{1}{x^2}dx = \frac{1}{100\pi} - \frac{1}{200\pi} < \frac{1}{100\pi}.$$

(ii) Lower bound: using integration by parts as in part (i),

$$\int_{100\pi}^{200\pi} \frac{\cos(x)}{x}dx = \int_{100\pi}^{200\pi} \frac{\sin(x)}{x^2}dx = \sum_{k=0}^{99} \int_{100\pi+k\pi}^{100\pi+(k+1)\pi} \frac{\sin(x)}{x^2}dx$$

$$= \sum_{p=0}^{49}\left(\int_{100\pi+2p\pi}^{100\pi+(2p+1)\pi} \frac{\sin(x)}{x^2}dx + \int_{100\pi+(2p+1)\pi}^{100\pi+(2p+2)\pi} \frac{\sin(x)}{x^2}dx\right)$$

$$= \sum_{p=0}^{49} \int_{100\pi+2p\pi}^{100\pi+(2p+1)\pi} \sin(x)\left(\frac{1}{x^2} - \frac{1}{(x+\pi)^2}\right)dx = \sum_{p=0}^{49} \int_{100\pi+2p\pi}^{100\pi+(2p+1)\pi} \frac{\pi\sin(x)}{x^2(x+\pi)^2}(2x+\pi)dx > 0,$$

as each of the integrals under the summation sign is positive (each of the corresponding integrands is a non-negative continuous function which is not identically equal to zero).

0.2 Uniform convergence

11. (Q.11) Uniform contunuity of $f : A \to \mathbb{R}$ means that for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x, x' \in A$: $|x - x'| < \delta$, $|f(x) - f(x')| < \epsilon$. To prove that f is continuous at any $x_0 \in A$, let us just set $x' = x_0$ in the above inequalities, which will then read as a standard definition of continuity at x_0.

12. (Q.12) (i) $f(x) = x^2$ is a polynomial, hence continuous on $\mathbb{R}$ (Analysis II). On the other hand, $|f(x+\delta/2) - f(x-\delta/2)| = 2|x||\delta| \to \infty$ for $x \to \infty$. Therefore, $|f(x+\delta/2) - f(x-\delta/2)|$ can be made as large as we like by varying x, no

matter how small δ is. Therefore, f is not uniformly continuous. (ii) Take for example $f(x) = 1/(1+x^2)$. Then for $\delta > 0$,

$$|f(x+\delta/2)-f(x-\delta/2)| = \delta\frac{2|x|}{(1+(x+\delta/2)^2)(1+(x-\delta/2)^2)} \leq \delta\frac{2|x|}{(1+2x^2)} \leq \frac{\delta}{\sqrt{2}},$$

where the last bound comes from a direct analysis of the function $f(R) = 2R/(1+2R^2)$ for $R \geq 0$. Therefore, f is uniformly continuous with $\delta_\epsilon = \frac{\epsilon}{\sqrt{2}}$.

13. (Q.13) Due to (i) all integrals over finite intervals exist. Fix any $\epsilon > 0$. As the integral $\int_a^\infty g < \infty$ (condition (ii)), there exists $b > a$ such that $\int_b^\infty g < \epsilon/3$. As $f_n \to f$ uniformly on $[a,b]$, there exists $N \in \mathbb{N}$ such that for any $n > N$ and any $x \in [a,b]$, $|f(x) - f_n(x)| < \epsilon/(3(b-a))$. Condition (ii) implies that $|f(x)| \leq g(x)$, $x \in [a,\infty)$. Therefore, for any $n > N$,

$$|\int_a^\infty f - \int_a^\infty f_n| \leq \int_a^b |f_n - f| + \int_b^\infty |f_n - f| \leq \epsilon/3 + 2\int_b^\infty g < \epsilon.$$

Therefore, $\lim_{n\to\infty} \int_a^\infty f_n = \int_a^\infty f$ by definition.

14. (Q.14) Fix $x > 1$. Let $f_n(t) = t^{x-1}(1-t/n)^n \mathbb{I}_{[0,n]}(t)$, $n \in \mathbb{N}$, $f(t) = g(t) = t^{x-1}e^{-t}$. Here $\mathbb{I}_{[0,n]}$ is the indicator function of the interval $[0,n]$: $\mathbb{I}_{[0,n]}(t) = 1$ if $t \in [0,n]$ and zero otherwise. Notice that all functions introduced above as continuous, hence regulated on $[0,R]$ for any $R > 0$. Moreover, $\int_0^n t^{x-1}(1 - t/n)^n dt = \int_0^\infty f_n$ and

$$f_n(t) = t^{x-1}e^{n\log(1-t/n)}\mathbb{I}_{[0,n]}(t) \leq t^{x-1}e^{-t} = g(t),\ t \geq 0,$$

where we used the inequality $\log(1+x) \leq x$, $x > 0$. Therefore g dominates the sequence (f_n). Notice that $\int_0^\infty g$ converges, this is just Euler's integral of the second kind, see Question 4. In order to apply the dominated convergence theorem it remains to prove that $f_n \to f$ uniformly on $[0,R]$. Accoridng to Taylor's theorem,

$$\log(1-y) = -y - \frac{1}{2}\frac{1}{(1-\xi)^2}y^2,\ y > 0,$$

where $\xi \in (0,y)$. Therefore,

$$||f_n - f||_\infty \leq R^{x-1}||e^{t+n\log(1-t/n)} - 1||_\infty = R^{x-1}||e^{-\frac{t^2}{2n}\frac{1}{(1-\xi)^2}} - 1||_\infty$$

$$\leq R^{x-1}|e^{-\frac{R^2}{2n}\frac{1}{(1-R/n)^2}} - 1| \to 0,\ n \to \infty.$$

Therefore, $f_n \to f$ uniformly on $[0,R]$. So we can use the dominated convergence theorem to claim that

$$\int_0^n t^{x-1}(1-t/n)^n dt = \int_0^\infty f_n \to \int_0^\infty f = \int_0^\infty t^{x-1}e^{-t}dt = \Gamma(x), n \to \infty.$$

15. (Q.15) (i) For $x \in [1+\delta,\infty)$, $1/k^x \leq (1/k)^{(1+\delta)}$, $k = 1,2,\ldots$. As the series $\sum_{k=1}^\infty 1/k^{1+\delta}$ converges, the zeta-function series converges uniformly on $[1+\delta,\infty)$ by the Weierstrass M-test. (ii) No, the zeta-function series does not converge uniformly on $(1,\infty)$. Really,

$$|\sum_{k=1}^n 1/k^x - \sum_{k=1}^\infty 1/k^x| = \sum_{k=n+1}^\infty 1/k^x \geq \int_n^\infty 1/y^x dy = \frac{n^{1-x}}{x-1}.$$

Choosing $x = 1 + 1/n$, we find that

$$|\sum_{k=1}^{n} 1/k^x - \sum_{k=1}^{\infty} 1/k^x| \geq n^{1+1/n} \geq n,$$

which can be made as large as we like by increasing n. This contradicts the definition of uniform convergence (if unsure, write down the negation of the definition).

16. (Q.16) We will prove that the sequence of partial sums is uniformly Cauchy. The sum of the n-th and the $(n+1)$-st terms of the series is

$$f_n(x)+f_{n+1}(x) = (-1)^n x^n \frac{\sqrt{n+1} - x\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = (-1)^n x^n \frac{(1-x^2)n+1}{\sqrt{n}\sqrt{n+1}(\sqrt{n+1}+x\sqrt{n})}.$$

For any $x \in [0,1]$, the absolute value of the above can is bounded by

$$\frac{x^n(1-x^2)n+1}{\sqrt{n}(n+1)}.$$

On the interval $[0,1]$, $x^n(1-x^2)$ is maximised at $x_c = \sqrt{n/(1+n)}$, which gives

$$||f_n(x) + f_{n+1}(x)||_\infty \leq \frac{2}{\sqrt{n}(n+1)}.$$

Therefore,

$$||\sum_{k=n}^{m} f_n(x)||_\infty \leq \sum_{k=n}^{\infty} \frac{2}{\sqrt{k}(k+1)}.$$

The right hand side of the above inequality vanishes in the limit $n \to \infty$, as the series $\sum_k \frac{2}{\sqrt{k}(k+1)}$ converges. Therefore, the sequence of partial sums is uniformly Cauchy and the series in question converges uniformly on $[0,1]$.

17. (Q.17) $f(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)}$. Let

$$f_n(x) = \sum_{k=1}^{n}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)},\ f_n'(x) = \sum_{k=1}^{n}(-1)^{k-1}x^{2k-2}.$$

Note that (f_n) and (f_n') converge uniformly by the M-test with $M_k = max(|a|,b)^{2k-1}$ and $M_k = max(|a|,b)^{2k-2}$ correspondingly. Therefore, the termwise differentiation is justified and

$$f'(x) = \lim_{n\to\infty} f_n'(x) = \frac{1}{1+x^2}.$$

Notice that $f(0) = 0$ and f' is continuous on $[a,b]$. By FTC2,

$$f(x) = \int_0^x f'(t)dt = \arctan(x).$$

18. (Q.18) The series $f(x) = \sum_{k=0}^{\infty}(k+1)x^k$ converges uniformly on $[a,b]$ by the Weierstrass M-test with $M_k = (k+1)max(-a,b)^k$. Therefore, f is continuous and the series can be integrated termwise:

$$\int_0^x f(t)dt = \sum_{k=0}^{\infty}\int_0^x (k+1)t^k dt = \sum_{k=0}^{\infty} x^{k+1} = 1 + \frac{1}{1-x}.$$

BY FTC1,

$$f(x) = \frac{d}{dx}\int_0^x f(t)dt = \frac{1}{(1-x)^2}.$$

19. (Q.19)The series $f(x) = \sum_{k=1}^{\infty} k(k+1)x^{k-1}$ converges uniformly on $[a,b]$ by the Weierstrass M-test with $M_k = k(k+1)max(-a,b)^k$. Therefore, f is continuous and the series can be integrated termwise:

$$\int_0^x f(t)dt = \sum_{k=1}^{\infty} \int_0^x k(k+1)t^{k-1}dt = \sum_{k=1}^{\infty}(k+1)x^k = -1 + \frac{1}{(1-x)^2},$$

where the last equality uses the result of Question 18. BY FTC1,

$$f(x) = \frac{d}{dx}\int_0^x f(t)dt = \frac{2}{(1-x)^3}.$$

20. (Q.20)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)3^{n-1}} = \sqrt{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)}\left(\frac{1}{\sqrt{3}}\right)^{2n-1} = \sqrt{3}\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^{1/\sqrt{3}} x^{2n-2}dx$$

$$= \sqrt{3}\int_0^{1/\sqrt{3}} \frac{1}{1+x^2}dx = \sqrt{3}\arctan(1/\sqrt{3}) = \sqrt{3}\cdot\frac{\pi}{6}.$$

The interchange of the order of summation and integration is justified, as $\sum_{n=0}^{\infty}(-y)^n$ converges uniformly in $[0,1/3]$.

November the 9th, 2015 Sergey Nazarenko and Oleg Zaboronski.