15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to two randomly chosen 'B' and two 'A' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' and 'C' questions.

The answers to ALL questions are to be submitted by the deadline of 3pm on Monday, the 1st of December 2014. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeon loft if you are a Maths student, or in the drop box labeled with your teaching assistant's name, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

0.1 Norms

1. *A. In our course we only study norms on real vector spaces. Let V be a* **complex** *vector space. A function $||\cdot|| : V \to \mathbb{R}$ is called a norm if:*

   - $||\mathbf{z}|| \geq \mathbf{0}$ *for all* $\mathbf{z} \in \mathbf{V}$
   - *If* $||\mathbf{z}|| = 0$*, then* $\mathbf{z} = \mathbf{0}$
   - *If* $\lambda \in \mathbb{C}$ *and* $\mathbf{z} \in V$*, then* $||\lambda\mathbf{z}|| = |\lambda| \cdot ||\mathbf{z}||$ *(Here $|\lambda|$ is the modulus of $\lambda \in \mathbb{C}$)*
   - *If* $\mathbf{z}, \mathbf{w} \in V$*, then* $||\mathbf{z} + \mathbf{w}|| \leq ||\mathbf{z}|| + ||\mathbf{w}||$.

   Which of the following functions are norms on $\mathbb{C}^n$?

   (a) $N_1(\mathbf{z}) = \sum_{k=1}^{n} (z_k + \bar{z}_k)$;

   (b) $N_2(\mathbf{z}) = \left(\sum_{k=1}^{n} \bar{z}_k z_k\right)^{1/2}$.

   (Here $\mathbf{z} = \{z_k = x_k + iy_k\}_{k=1}^n, \bar{\mathbf{z}} = \{x_k - iy_k\}_{k=1}^n \in \mathbb{C}^n$). Justify your answers

2. *A. A pair (X, d) is called a metric space if X is a set and $d : X \times X \to \mathbb{R}$ is a function with the following properties: for any $x, y, z \in X$,*

   - $d(x, y) \geq 0$ *(non-negativity)*
   - $d(x, y) = 0$ *iff* $x = y$ *(separation of points)*
   - $d(x, y) = d(y, x)$ *(symmetry)*
   - $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

   Let $(V, ||\cdot||)$ be a normed vector space. Define $d(x, y) := ||x - y||,\ x, y \in V$. Prove that (V, d) is a metric space.

3. *A. Let $a_j \in \mathbb{R}$ for $1 \leq j \leq n$ and write*

$$||x||_a = \sum_{j=1}^{n} a_j |x_j|.$$

State and prove necessary and sufficient conditions for $||\cdot||_a$ to be a norm on $\mathbb{R}^n$.

4. *B. Consider S_F, the space of real sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$, such that* **all but finitely many** *of the a_n's are zero. (In other words, each sequence $\mathbf{a} \in S_F$ is eventually zero.)*

   (a) *Show that if we use the natural definition of addition and scalar multiplication*

   $$(a_n) + (b_n) = (a_n + b_n),\ \lambda(a_n) = (\lambda a_n),\ \lambda \in \mathbb{R},$$

   then S_F is a vector space.

   (b) *Show that the following definitions all give norms on S_F,*

   $$||\mathbf{a}||_\infty = \max_{n\geq 1} |a_n|, \tag{1}$$
   $$||\mathbf{a}||_w = \max_{n\geq 1} |na_n|, \tag{2}$$
   $$||\mathbf{a}||_1 = \sum_{n=1}^{\infty} |a_n|, \tag{3}$$
   $$||\mathbf{a}||_2 = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}, \tag{4}$$
   $$||\mathbf{a}||_u = \sum_{n=1}^{\infty} n|a_n|. \tag{5}$$

   (c) *Show that norms (2), (3), (4), (5) are NOT equivalent to norm (1). (In fact no two of norms from the above list are equivalent. Verify this if you accidentally got a free Saturday night. At least, check the most interesting case of (2) and (3).)*

0.2 Completeness.

5. *A. Let S_F be the space defined in Question 4. Prove that the normed vector space $(S_F, ||\cdot||_1)$ is not Banach. Here $||\cdot||_1$ is the norm defined in (3).*

6. *B. Let l_1 be the set of real sequences $\mathbf{a}$ with $\sum_{j=1}^{\infty} |a_j|$ convergent.*

   (a) *Show that l_1 is a vector space given the natural definitions of addition and multiplications:*

   $$(a_n) + (b_n) = (a_n + b_n),\ \lambda(a_n) = (\lambda a_n),\ \lambda \in \mathbb{R}.$$

   (b) *Prove that $(l_1, ||\cdot||_1)$ is a complete normed vector space (Banach space). Here $||\cdot||_1$ is the function defined in (3). You may assume without a proof that $||\cdot||_1$ is a norm on l_1.*

7. *A. A subset X of a normed vector space $(V, ||\cdot||)$ is said to possess the Bolzano-Weierstrass property if every sequence $(x_n)_{n=1}^{\infty} \subset X$ has a convergent subsequence $x_{n_k} \overset{k\to\infty}{\longrightarrow} x \in X$. Prove that X is complete in the sense that every Cauchy sequence in X converges to a point in X.*

8. *A. This question is a preparation for the discussion of the Banach contraction mapping theorem. Let* $(X, ||\cdot||)$ *be a normed vector space and* $T : X \to X$ *a mapping. We say that* $w \in X$ *is a fixed point of* T *if* $Tw = w$*. We say that* T *is a contraction mapping if there exists a positive number* $K < 1$ *with* $||Tx - Ty|| \leq K||x - y||$ *for all* $x, y \in X$*. Suppose that* $x_0 \in X$ *and define* x_n *inductively by* $x_n = Tx_{n-1}$*,* $n > 0$*. Show that the sequence* $(x_n)_{n\geq 1}$ *converges to* w *for any* $x_0 \in X$*.*

0.3 Closed and open sets, continuity

9. *B. Consider the real normed vector space* l^∞ *of bounded real sequences* $\mathbf{a} = (a_1, a_2, \ldots)$ *with norm* $||a||_\infty = sup_{n\geq 1}|a_n|$*. The linear structure is given by component-wise addition and scalar multiplication defined in Question 4(a).*

   (a) *Show that the set*
   $$E = \{\mathbf{a} \in l^\infty : \exists N(\mathbf{a}) : \ \forall n > N(\mathbf{a}), a_n = 0\}.$$
   is a linear subspace of l^∞*, but not a closed subset.*

   (b) *Show that the set*
   $$F = \{\mathbf{a} \in l^\infty : a_{2n} = 0 \ \forall n = 1, 2, 3, \ldots\}.$$
   is a closed infinite-dimensional linear subspace of l^∞*.*

   (c) *Show that any finite-dimensional linear subspace of any normed vector space is closed.*

10. *B. Consider the vector space* S_F *defined in Question 4.*

    (a) *Show that the map* $T : S_F \to \mathbb{R}$ *defined by* $T\mathbf{a} = \sum_{j=1}^\infty a_j$ *is linear.*

    (b) *B. If we equip* $\mathbb{R}$ *with the usual Euclidean norm* $|\cdot|$ *and* S_F *with one of the five norms (1)-(5), state, with reasons, whether* T *is continuous.*

11. *C. Let* $U = V = S_F$*, where* S_F *a normed space defined in Question 4. Let* $||\mathbf{a}||_V = ||\mathbf{a}||_U = \sum_{n=1}^\infty |a_n|$*.*

    (a) *Show that if* $T : U \to V$ *is defined by* $T\mathbf{a} = \mathbf{b}$ *with* $b_j = (1 - j^{-1})a_j$*,* $j = 1, 2, 3, \ldots$*, then* T *is a continuous linear map. However, prove that there does not exist an* $\mathbf{a} \in U$ *with* $a \neq 0$ *such that* $||T\mathbf{a}||_V = ||T||||\mathbf{a}||_U$*.*

    (b) *If* U *and* V *are two finite-dimensional normed vector spaces and* $T : U \to V$ *is linear, can we always find an* $\mathbf{a} \in U$ *with* $\mathbf{a} \neq 0$ *such that* $||T\mathbf{a}||_V = ||T||||\mathbf{a}||_U$*? Give reasons.*

17th of November 2014 *Daniel Ueltschi and Oleg Zaboronski*