**Questions for credit:** 2 (4 points), 8 (5 points), 4 (8 points) and 10 (8 points)

## 0.1 Norms

1. (a) $N_1(\mathbf{z})$ is not a norm on $\mathbb{C}^n$. For example, it does not separate points: take $\mathbf{z} = (i, i, \ldots, i)$. Then $\bar{\mathbf{z}} = (-i, -i, \ldots, -i)$. So, $z_k + \bar{z}_k = 0$ for all $k$ and $N_1(\mathbf{z}) = 0$ even though $\mathbf{z} \neq 0$.

   (b) $N_2(z)$ is a norm on $\mathbb{C}^n$. Positivity: $z_k\bar{z}_k = |z|_k^2 \geq 0$, so $\sum_{k=1}^n z_k\bar{z}_k \geq 0$. Separation of points: if $N_2(\mathbf{z}) = 0$, then $z_k\bar{z}_k = |z_k|^2 = 0$ for $k = 1, 2, \ldots n$, as each term in the sum for $N_2(\mathbf{z})^2$ is non-negative. Therefore, $z_k = 0$ for $k = 1, 2, \ldots, n$, meaning that $\mathbf{z} = 0$. Absolute homogeneity: $N_2(\lambda\mathbf{z}) = (\sum_{k=1}^n \lambda\bar{\lambda}z_k\bar{z}_k)^{1/2} = |\lambda|N_2(\mathbf{z})$. Triangle inequality: $N_2(\mathbf{w}+\mathbf{z}) = (\sum_{k=1}^n |z_k+w_k|^2)^{1/2} = (\sum_{k=1}^n((Re(z_k)+Re(w_k))^2+(Im(z_k)+Im(w_k))^2))^{1/2} \leq (\sum_{k=1}^n(Re(z_k)^2+Im(z_k)^2))^{1/2}+(\sum_{k=1}^n(Re(w_k)^2+Im(w_k)^2))^{1/2} = (\sum_{k=1}^n |z|_k^2)^{1/2} + (\sum_{k=1}^n |w|_k^2)^{1/2} = N_2(\mathbf{z}) + N_2(\mathbf{w})$. The inequality in the middle is the triangle inequality for the **Euclidean** norm on $\mathbb{R}^{2n}$.

2. Non-negativity: as $x - y \in V$ and $||\cdot||$ is a norm on $V$, $d(x, y) = ||x - y|| \geq 0$. Separation of points: if $d(x, y) = ||x - y|| = 0$, then $x - y = 0$, as the norm $||\cdot||$ separates points. Therefore $x = y$. Symmetry: $d(x-y) = ||x-y|| = ||(-1)(y-x)|| = |-1| \cdot ||y - x|| = d(y, x)$, where we used the absolute homogeneity property of $||\cdot||$. Triangle inequality: $d(x, z) = ||x - z|| = ||x - y + y - z|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$, where we used the triangle inequality for $||\cdot||$.

   Note that there is a significant difference between normed and metric spaces: the definition of the latter does not require the space to be linear!

3. **Claim**: $||\cdot||$ is a norm on $\mathbb{R}^n$ iff $a_i > 0,\ i = 1, 2, \ldots n$.
   **Proof**. $\Rightarrow$ $||\cdot||_a$ is a norm, therefore for every $i = 1, 2, \ldots, n$,

$$||(0, 0, \ldots, 0, \overset{i\text{-th co-ordinate}}{1}, 0, \ldots 0)||_a > 0.$$

   Therefore, $a_i > 0$.

   $\Leftarrow$ If $a_i > 0,\ i = 1, 2, \ldots n$, the function $||\cdot||_a$ clearly separates points, is positive on $\mathbb{R}^n \setminus \{0\}$ and is absolutely homogeneous. Triangle inequality: $||x + y||_a = \sum_{k=1}^n a_k|x_k + y_k| \leq \sum_{k=1}^n a_k(|x_k| + |y_k|) = ||x||_a + ||y||_b$.

4. (a) The most important step is to verify that $S_F$ is closed under addition and scalar multiplication: for any $\mathbf{a}, \mathbf{b} \in S_F$ there exist $N_1, N_2 \in \mathbb{N}$: $a_n = 0$ for any $n > N_1$, $b_n = 0$ for any $n > N_2$. Let $N = max(N_1, N_2)$. Then $(a_n + b_n) = 0$ for $n > N$. Therefore, $\mathbf{a} + \mathbf{b} \in S_F$. Similarly, for any $\lambda \in \mathbb{R}$, $\lambda a_n =$ for $n > N_1$, so $\lambda\mathbf{a} \in S_F$.

   The verification of the axioms of vector space ($S_F$ is an abelian group with respect to addition and $S_F$ is an $\mathbb{R}$-module) is elementary and students can be excused for going through it quickly.

   (b) The most economical way of proving that functions (1), (3), (4) and (5) are norms on $S_F$ is as follows: notice that the set $S_F(N) = \{\mathbf{a} \in S_F : a_n = 0,\ n > N\}$ is a linear subset of $S_F$ isomorphic to $\mathbb{R}^N$. The restriction of functions (1), (3), (4) and (5) to $S_F(N)$ are just the sup-norm, the taxicab norm, the Euclidean norm and the norm from Question 4 with $a_i = i > 0$ on $\mathbb{R}^N$ correspondingly. As for any pair of $\mathbf{a}, \mathbf{b} \in S_F$ there exists $N$: $\mathbf{a}, \mathbf{b} \in S_F$, the check that functions (1), (3), (4) and (5) are norms on $S_F$ is equivalent to checking that they give rise to norms on $S_F(N) \simeq \mathbb{R}$ for any $N \in \mathbb{N}$. This was done in the lectures for norms (1), (3), (4) and in Question 4 for norm (5).

To check that (2) is a norm on $S_F$, notice that it is well defined as the max is always taken over a finite set. The separation of points and absolute homogeneity properties are straightforward. Triangle inequality: $||\mathbf{a}+\mathbf{b}||_w = \max_{n\geq 1} n|a_n + b_n| \leq \max_{n\geq 1} n(|a_n| + |b_n|) \leq \max_{n\geq 1} n|a_n| + \max_{n\geq 1} n|b_n| = ||\mathbf{a}||_w + ||\mathbf{b}||_w$.

(c) (i) (2)$\not\sim$(1). Let $(\mathbf{a}_n)_{n\geq 1} \subset S_F$ be a sequence:

$$\mathbf{a}_n = (\underbrace{1, 1, \ldots, 1}_{n\text{-times}}, 0, 0 \ldots).$$

For any $n \in \mathbb{N}$, $||\mathbf{a}_n||_\infty = 1$, $||\mathbf{a}_n||_w = n \to \infty$ for $n \to \infty$. Therefore $\nexists$ $K_1 > 0$: for any $n \in \mathbb{N}$, $K_1||\mathbf{a}_n||_w < ||\mathbf{a}_n||_\infty$, meaning that $||\cdot||_w \not\sim ||\cdot||_\infty$.

(ii) (3)$\not\sim$(1). Using the sequence $(\mathbf{a}_n)_{n>1}$ from the previous step we see that $||\mathbf{a}_n||_1 = n \to \infty$ as $n \to \infty$. Therefore, (3)$\not\sim$(1) by the argument given in the previous step.

(iii) (4)$\not\sim$(1). Using the sequence $(\mathbf{a}_n)_{n>1}$ from step (i) we see that $||\mathbf{a}_n||_2 = n^{1/2} \to \infty$ as $n \to \infty$. Therefore, (4)$\not\sim$(1) by the argument given in step (i).

(iv) (5)$\not\sim$(1). Using the sequence $(\mathbf{a}_n)_{n>1}$ from step (i) we see that $||\mathbf{a}_n||_u = n(n+1)/2 \to \infty$ as $n \to \infty$. Therefore, (5)$\not\sim$(1) by the argument given in step (i).

## 0.2 Completeness

5. Let $(\mathbf{a}_n)_{n\geq 1} \subset S_F$ be a sequence such that

$$(\mathbf{a}_n)_k = \begin{cases} 2^{-k} & \text{if } 1 \leq k \leq n; \\ 0 & \text{if } k > n. \end{cases}$$

For any pair of natural numbers $k < m$,

$$0 \leq ||\mathbf{a}_m - \mathbf{a}_k||_1 = \sum_{p=k+1}^{m} 2^{-p} \leq 2^{-k} \to 0 \text{ for } k \to \infty.$$

Therefore, the sequence $(\mathbf{a}_n)_{n\geq 1}$ is $||\cdot||_1$-Cauchy. However, $(\mathbf{a}_n)_{n\geq 1}$ does not converge in $S_F$ for suppose that $\mathbf{a} \in S_F$ is the limit. Then there is $N \in \mathbb{N}$ such that $a_N = 0$. So, for any $n > N$ $||\mathbf{a}_n - \mathbf{a}||_1 \geq (\mathbf{a}_n)_N = 2^{-N}$. Hence for $\epsilon = 2^{-N-1}$ and for any $M \in \mathbb{N}$, there exists $n > M$ such that $||\mathbf{a}_n - \mathbf{a}||_1 > \epsilon$, which contradicts our assumption that $\mathbf{a}$ is the limit.

We have constructed a Cauchy sequence in $S_F$ which does not converge. Therefore, $S_F$ is not Banach.

6. (a) The crucial step is the proof that $l_1$ is closed with respect to component-wise addition. This is true as the sum of two absolutely convergent series is absolutely convergent: $\sum_{n=1}^{\infty}|a_n+b_n| \leq \sum_{n=1}^{\infty}(|a_n|+|b_n|) \leq \infty$. The rest is a straightforward consequence of theorems governing operations on convergent series.

(b) Let $(\mathbf{a}_n)_{n\geq 1} \subset l_1$ be a Cauchy sequence. Then $\forall \epsilon > 0$, there exists $N_\epsilon$ : for any $m > n > N_\epsilon$,

$$||\mathbf{a}_m - \mathbf{a}_n||_1 \equiv \sum_{k=1}^{\infty} |(\mathbf{a}_m)_k - (\mathbf{a}_n)_k| < \epsilon/2 \tag{1}$$

Therefore, for any $k \in \mathbb{N}$, $|(\mathbf{a}_m)_k - (\mathbf{a}_n)_k| < \epsilon$. We proved that the sequence $((\mathbf{a}_n)_k)_{n\geq 1}$ is Cauchy in $(\mathbb{R}, |\cdot|)$. Therefore it converges. Let $a_k$ be the limit. Consider the sequence $\mathbf{a} = (a_1, a_2, a_3, \ldots)$.

We proved that $\mathbf{a}_n \to \mathbf{a}$ component-wise. Now we need to prove that (i) $\mathbf{a}_n \to \mathbf{a}$ w. r. t. $||\cdot||_1$ norm; (ii) $\mathbf{a} \in l_1$. Both results are a consequence of the uniform convergence of the series $\sum_{k=1}^{\infty}(\mathbf{a}_n)_k$ with respect to $n$:

(i) Using (1): for any $M \in \mathbb{N}$, and any $m > n > N_\epsilon$, $\sum_{k=1}^{M}|(\mathbf{a}_m)_k - (\mathbf{a}_n)_k| \leq ||\mathbf{a}_m - \mathbf{a}_n||_1 < \epsilon/2$. Taking the limit $m \to \infty$ we find

$$\sum_{k=1}^{M} |a_k - (\mathbf{a}_n)_k| \leq \epsilon/2.$$

Now we can take the limit $M \to \infty$:

$$||\mathbf{a} - \mathbf{a}_n||_1 \leq \epsilon/2.$$

We proved that for any $\epsilon > 0$, there is $N_\epsilon$ such that for any $n > N_\epsilon$, $||\mathbf{a} - \mathbf{a}_n||_1 < \epsilon$. Therefore, $(\mathbf{a}_n)_{n\geq 1}$ converges to $\mathbf{a}$ with respect to $||\cdot||_1$ norm.

(ii) It also follows from (1) that

$$0 \leq \sum_{k=1}^{M} |(\mathbf{a}_m)_k| \leq ||\mathbf{a}_m||_1 \leq \epsilon/2 + ||\mathbf{a}_n||_1.$$

Taking the limit $m \to \infty$ in the above equation we conclude that for any $M \in \mathbb{N}$, $\sum_{k=1}^{M}|a_k| \leq \epsilon/2 + ||\mathbf{a}_n||_1$. Therefore the series $\sum_{k=1}^{\infty}|a_k|$ converges and $\mathbf{a} \in S_F$.

We proved that any $||\cdot||_1$-Cauchy series in $l_1$ converges. Therefore, $(l_1, ||\cdot||_1)$ is Banach.

7. Let $(x_k)_{k\geq 1} \subset X$ be a Cauchy sequence, $(x_{k_p})_{p\geq 1} \subset (x_k)_{k\geq 1}$ - its convergent subsequence, which exists by the BW property of $X$. Let $x \in X$ be the limit of $(x_{k_p})_{p\geq 1}$. Thus for any $\epsilon > 0$, there exists $N_\epsilon$ : (i) For any $p : k_p > N_\epsilon$, $||x_{k_p} - x|| < \epsilon/2$ (convergence of $(x_{k_p})$); (ii) For any $m, n > N_\epsilon$, $||x_m - x_n|| < \epsilon/2$ (Cauchy property of $(x_k)$). Therefore, for any $\epsilon > 0$, there exists $N_\epsilon$: for any $n > N_\epsilon$, $||x_n - x|| \leq ||x_n - x_{k_p}|| + ||x_{k_p} - x|| < \epsilon/2 + \epsilon/2 < \epsilon$. So $x_n \to x$ by definition. We proved that every Cauchy sequence in $X$ converges. In other words, $X$ is complete.

8. Let $n > 0$. $||x_n - w|| = ||Tx_{n-1} - Tw|| \leq K||x_{n-2} - w||$, where we used $Tw = w$. We proved that for any $n > 0$,

$$||x_n - w|| \leq K||x_{n-1} - w||.$$

Iterating the above inequality $n$ times we find

$$||x_n - w|| \leq K^n||x_0 - w|| \overset{n\to\infty}{\longrightarrow} 0,$$

as $0 < K < 1$. So $x_n \to w$ by definition.

## 0.3 Closed and opened sets, continuity

9. (a) Noticing that $E \cong S_F$, we can immediately conclude (Question 4(a)) that $E \in l^\infty$ is a linear subspace. It is not closed: Let $(\mathbf{a}_n)_{n\geq 1} \subset E$ be a sequence such that

$$(\mathbf{a}_n)_k = \begin{cases} 2^{-k} & \text{if } 1 \leq k \leq n; \\ 0 & \text{if } k > n. \end{cases}$$

This sequence converges to $\mathbf{a} = (2^{-k})_{k\geq 1} \in l^\infty$:

$$||\mathbf{a}_n - \mathbf{a}||_\infty = 2^{-n-1} \overset{n\to\infty}{\longrightarrow} 0.$$

Notice that $\mathbf{a} \notin E$ as $(\mathbf{a})_k = 2^{-k} > 0$ for any $k \in \mathbb{N}$. We found a convergent sequence contained in $E$ which has a limit in $l^\infty \setminus E$. Therefore, $E$ is not closed by our Proposition 45.

(b) The check that $F$ is closed with respect component-wise addition and scalar multiplication is elementary. Let $(\mathbf{a}_n)_{n\geq 1} \subset F$ be a convergent subsequence. Let $\mathbf{a} \in l^\infty$ be the limit. Then for any $p \in \mathbb{N}$,

$$0 \leq |(\mathbf{a})_{2p}| = |(\mathbf{a})_{2p} - (\mathbf{a}_n)_{2p}| \leq sup_{k\geq 1}|(\mathbf{a})_k - (\mathbf{a}_n)_k| = ||\mathbf{a} - \mathbf{a}_n||_\infty \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore, $(\mathbf{a})_{2p} = 0$ for all $p \in \mathbb{N}$ and $\mathbf{a} \in F$. We proved that any convergent sequence in $F$ converges to an element of $F$. Therefore $F$ is closed by Proposition 45.

(c) Let $(U, ||\cdot||$ be a normed linear space, $V \subset U$ - a finite dimensional linear subspace. Then $(V, ||\cdot||)$ is a finite dimensional normed vector space with respect to the restriction of the norm $||\cdot||$ to V. We know that $(V, ||\cdot||)$ is Banach (Theorem 37). Let $(x_n)_{n\geq 1} \subset V$ be a convergent sequence in $V$. Any convergent sequence is Cauchy, therefore the limit must belong to $V$ due to the completeness of the latter. We proved that every convergent subsequence in $V$ converges to an element of $V$, so $V$ is closed by Proposition 45.

10. (a) For any $\mathbf{a}, \mathbf{b} \in S_F$ and $\lambda, \mu \in \mathbb{R}$, $T(\lambda\mathbf{a} + \mu\mathbf{b}) = \sum_{k=1}^N(\mu a_k + \lambda b_k) = \mu\sum_{k=1}^N a_k + \lambda\sum_{k=1}^N b_k = \mu T(\mathbf{a}) + \lambda T(\mathbf{b})$, so the map is linear. In the derivation, $N \in \mathbb{N} : a_k = 0, b_k = 0$ for any $k > N$. It exists as $\mathbf{a}, \mathbf{b} \in S_F$.

(b) Here we apply Theorem 39 of the lecture notes:
(i) Norm (1). Let $(\mathbf{a}_n)_{n\geq 1} \subset S_F$ be a sequence:

$$\mathbf{a}_n = (\underbrace{1, 1, \ldots, 1}_{n\text{-times}}, 0, 0\ldots).$$

For any $n$, $||a_n||_\infty = 1$, but $T(\mathbf{a}_n) = n \to \infty$ as $n \to \infty$. Therefore, $T$ is not bounded, hence NOT continuous.

(ii) Norm (2). Let $(\mathbf{a}_n)_{n\geq 1} \subset S_F$ be a sequence:

$$\mathbf{a}_n = (1, 1/2, \ldots, 1/n, 0, 0 \ldots).$$

For any $n$, $||a_n||_w = 1$, but $T(\mathbf{a}_n) = \sum_{k=1}^{n}(1/k) \to \infty$ as $n \to \infty$. Therefore, $T$ is not bounded, hence NOT continuous.

(iii) Norm (3). Consider any $\mathbf{a} \in S_F : ||\mathbf{a}||_1 \leq 1$. This means that $\sum_{k=1}^{\infty} |a_k| \leq 1$. Therefore, $|T(\mathbf{a})| = |\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k| \leq 1$. We proved that $T$ is bounded, so it IS continuous.

(iv) Norm (4). Let $(\mathbf{a}_n)_{n\geq 1} \subset S_F$ be a sequence:

$$\mathbf{a}_n = \frac{1}{Z}(1, 1/2, \ldots, 1/n, 0, 0 \ldots),$$

where $Z = \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2} < \infty$. For any $n$,

$$||\mathbf{a}_n||_2 = \frac{\left(\sum_{k=1}^{n} \frac{1}{k^2}\right)^{1/2}}{\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2}} \leq 1,$$

but $T(a_n) = \frac{1}{Z}\sum_{k=1}^{n}(1/k) \to \infty$ as $n \to \infty$. Therefore, $T$ is not bounded, hence NOT continuous.

(v) Norm (5). Consider any $\mathbf{a} \in S_F : ||\mathbf{a}||_u \leq 1$. This means that $\sum_{k=1}^{\infty} k|a_k| \leq 1$. Therefore, $|T(\mathbf{a})| = |\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} k|a_k| \leq 1$. We proved that $T$ is bounded, so it IS continuous.

11. (a) Linearity: For any $j \in \mathbb{N}$, $T(\mu\mathbf{a}) + \lambda\mathbf{b})_j = (1 - j^{-1})(\mu a_j + \lambda b_j) = \mu T(\mathbf{a})_j + \lambda T(\mathbf{b})_j$.

Continuity: Consider any $\mathbf{a} \in S_F : ||\mathbf{a}||_U \leq 1$. This means that $\sum_{k=1}^{\infty} |a_k| \leq 1$. Therefore, $||T(\mathbf{a})||_V = |\sum_{k=1}^{\infty}(1 - k^{-1})a_k| \leq \sum_{k=1}^{\infty} |a_k| \leq 1$. We proved that $T : U \to V$ is bounded, so it is continuous by our Thm 39.

A byproduct of the continuity proof is that $||T|| \leq 1$. Let $(\mathbf{a}_n)_{n\geq 1} \subset S_F$ be a sequence:

$$\mathbf{a}_n = (\underbrace{1, 1, \ldots, 1}_{n\text{-times}}, 0, 0 \ldots).$$

Then

$$||T\mathbf{a}_n||_1/||\mathbf{a}_n||_1 = \frac{(n - \sum_{k=1}^{n}(1/k))}{n} \to 1, \; n \to \infty.$$

Therefore, $||T|| = 1$ by the definition of *sup*. On other hand, for any $\mathbf{b} \in S_F \setminus \{\mathbf{0}\}$,

$$||T(\mathbf{b})||_1 = \sum_{k=1}^{\infty}(1 - k^{-1})|b_k| < ||\mathbf{b}||_1.$$

So there exists no non-zero element $\mathbf{a}$ of $S_F$ such that $||T\mathbf{a}||_V = ||T||||\mathbf{a}||_U$.

(b) Yes. Here is the sketch of the argument: using the definition of *sup* in the operator norm, we can construct a sequence $(x_n)_{n\geq 1} \subset U$, such that for any $n \in \mathbb{N}$, $||x_n||_U = 1$ and $||T(x_n)||_V \to ||T||$ as $n \to \infty$. As we proved in the lecture, any bounded sequence in a finite dimensional normed space has a convergent subsequence. Let $(x_{n_k})_{k\geq 1}$ be such a subsequence. Let $x \in U$ be its limit. Then,

$$0 \leq |||Tx_{n_k}||_V - ||Tx||_V| \leq ||T(x_{n_k} - x)||_V \leq ||T||||x_{n_k} - x||_U \to 0, \ k \to \infty.$$

Therefore, $||Tx||_V = ||T|| = ||T||||x||_U$.

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