

15% of the credit for this module will come from your work on four assignments submitted by a 3pm deadline on the Monday in weeks 4,6,8,10. Each assignment will be marked out of 25 for answers to **three** randomly chosen 'B' and **one** 'A' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the problems will be based on 'B' questions.

The answers to **ALL** questions are to be submitted by the deadline of **3pm on Monday, the 7th of December 2015**. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your supervisor or your teaching assistant. Your work should be deposited in your supervisor's slot in the pigeonloft if you are a Maths student, or in the dropbox labelled with the course's code, opposite the Maths Undergraduate Office, if you are a non-Maths or a visiting student.

0.1 Uniform convergence.

1. B. The purpose of this question is to investigate the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2} := \sum_{n=-\infty}^N \frac{1}{(x-n)^2} + \sum_{n=N+1}^{\infty} \frac{1}{(x-n)^2},$$

where N is any integer.

(a) Let M be a strictly positive integer. Show that the series

$$\sum_{n=M}^{\infty} \frac{1}{(x-n)^2} \quad \text{and} \quad \sum_{n=M}^{\infty} \frac{1}{(x+n)^2}$$

converge uniformly for $|x| \leq M/2$. Conclude that the series

$$\sum_{n=M}^{\infty} \left(\frac{1}{(x-n)^2} + \frac{1}{(x+n)^2} \right)$$

converges uniformly on $[-M/2, M/2]$. Let the limit be f_M . Prove that f_M is continuous on $[-M/2, M/2]$ and differentiable on $(-M/2, M/2)$.

(b) Use the result of part (a) to show that the function

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2}$$

is well defined, continuous and differentiable on $\mathbb{R} \setminus \mathbb{Z}$. **Hint.** Use an appropriate (possibly x -dependent) decomposition of the above series as the sum of two series from part (a). Show that $F(x+1) = F(x)$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$.

(c) (i) Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function: $g(x) = g(x+1)$ for all $x \in \mathbb{R}$. Prove that g is bounded. (ii) Let f be a bounded function on \mathbb{R} such that

$$f(x) = \frac{1}{4} \left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) \right)$$

for all x . Prove that $f(x) = 0$ for all x .

- (d) (i) Show by L'Hopital's rule that $\pi^2 \operatorname{cosec}^2(\pi x) - \frac{1}{x^2} \rightarrow \pi^2/3$ as $x \rightarrow 0$. (ii) Let F, f_M be functions defined in part (b) and (a) respectively. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as follows:

$$g(x) = F(x) - \pi^2 \operatorname{cosec}^2(\pi x) = f_1(x) - \left(\pi^2 \operatorname{cosec}^2(\pi x) - \frac{1}{x^2} \right), \quad x \in \mathbb{R} \setminus \mathbb{Z},$$

$g(x) = f_1(0) - \pi^2/3, x \in \mathbb{Z}$. Prove that g is continuous on \mathbb{R} and $g(x) = g(x+1)$ for all $x \in \mathbb{R}$.

- (e) Bringing everything together: consider function g defined in part (d). (i) Prove that it is bounded and satisfies the equation of part (c). Thus prove that $g \equiv 0$ and so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2} = \pi^2 \operatorname{cosec}^2(\pi x)$$

for all $x \in \mathbb{R} \setminus \mathbb{Z}$. (ii) Show that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$.

0.2 Norms.

2. B. Let $f \in C[0,1]$. Recall two of the norms we considered in class:

$$\|f\|_{\infty} = \sup_{t \in [0,1]} |f(t)|, \quad \|f\|_1 = \int_0^1 |f(t)| dt.$$

Consider the space $C^1([0,1])$ of continuously differentiable functions on $[0,1]$. Define

$$\|f\|_A = \|f\|_{\infty} + \|f\|_1, \quad (1)$$

$$\|f\|_B = \|f'\|_{\infty}, \quad (2)$$

$$\|f\|_C = \|f\|_{\infty} + \|f'\|_{\infty}, \quad (3)$$

$$\|f\|_D = |f(0)| + \|f'\|_1. \quad (4)$$

(i) Which of these formulae define norms? (ii) Consider the set of norms you found in part (i) together with the norms $\|\cdot\|_1, \|\cdot\|_{\infty}$. Identify the equivalence classes of these norms with respect to our definition of equivalence.

3. B. (Hölder's inequality) Let $p, q > 1$ be real numbers such that $p^{-1} + q^{-1} = 1$.

(a) Prove that for any $x > 0, y > 0$,

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy. \quad (5)$$

Check that the above inequality remains true if $x \geq 0, y \geq 0$.

Hint. Recall that a function $f : (a, b) \rightarrow \mathbb{R}$ is called convex if for any $x_1, x_2 \in (a, b)$ and any $\lambda \in [0, 1]$,

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2).$$

Derive (5) from the statement of convexity of $\log(1/x)$ on $(0, \infty)$. (You do not have to check that $\log(1/x)$ is convex.)

(b) Let $f, g \in C[a, b]$. Use (5) to prove that

$$\int_a^b |fg| \leq \frac{1}{p} \int_a^b |f|^p + \frac{1}{q} \int_a^b |g|^q.$$

Deduce that if $F, G \in C[a, b]$ are such that $\int_a^b |F|^p = \int_a^b |G|^q = 1$, then $\int_a^b |FG| \leq 1$.

(c) Use results of (b) to prove that for any $f, g \in C[a, b]$,

$$\int_a^b |fg| \leq \left(\int_a^b |f|^p \right)^{1/p} \left(\int_a^b |g|^q \right)^{1/q}.$$

(Hölder's inequality for integrals.) Estimate $\int_a^b fg$ above using Hölder inequality with $p = q = 2$. This is called Cauchy-Schwarz inequality.

4. B. (L_p spaces.) Define $p > 1, q > 1$ as in Question 3.

(a) Let $f \in C[a, b]$. Suppose there exists $A \in \mathbb{R}$:

$$\int_a^b |fg| \leq A \left(\int_a^b |g|^q \right)^{1/q}$$

for all continuous functions $g: [a, b] \rightarrow \mathbb{R}$. By taking $g = |f|^\alpha$ for a suitable α show that

$$\left(\int_a^b |f|^p \right)^{1/p} \leq A.$$

(b) Applying Hölder's inequality and then the inequality of part (a) to the integral bound

$$\int_a^b |fg + hg| \leq \int_a^b |fg| + \int_a^b |gh|,$$

show that

$$\left(\int_a^b |f + h|^p \right)^{1/p} \leq \left(\int_a^b |f|^p \right)^{1/p} + \left(\int_a^b |h|^p \right)^{1/p}$$

(Minkowski inequality.)

(c) Show that if we set

$$\|f\|_p = \left(\int_a^b |f|^p \right)^{1/p},$$

then $(C[a, b], \|\cdot\|_p)$ is a normed space (called L_p).

Remark. You have just proved that L_p is a normed space for $p > 1$. L_1 is also a normed space, as proved in the lectures.

0.3 Completeness.

5. B. Consider functions on $C^1[0, 1]$ defined by (1), (2), (3), (4) of Question 2 which you proved to be norms. Which of these norms are complete? (Jargon: A norm $\|\cdot\|$ on V is complete if $(V, \|\cdot\|)$ is a complete normed space.) **Hint.** To simplify the proof use the following statement mentioned in the lectures: if $\|\cdot\| \sim \|\cdot\|'$, then $(V, \|\cdot\|)$ is Banach iff $(V, \|\cdot\|')$ is Banach.

6. A. Let s_{00} be the space of real sequences $(a_n)_{n \geq 1}$ such that all but finitely many a_n 's are zero. Check that s_{00} is closed under component-wise addition and multiplication by scalars. Let $\|\cdot\|_1 : s_{00} \rightarrow \mathbb{R}$: for any $(a_n)_{n \geq 1} \in s_{00}$,

$$\|(a_n)_{n \geq 1}\|_1 = \sum_{n=1}^{\infty} |a_n|.$$

Check that this function is a norm on s_{00} . Prove that the space $(s_{00}, \|\cdot\|_1)$ is not complete.

Remark. For enthusiasts: prove that there exists no norm $\|\cdot\|$ on s_{00} such that $(s_{00}, \|\cdot\|)$ is complete.

7. B. Show that the space $L_p = (C[a, b], \|\cdot\|_p)$ studied in Question 4 is not complete for any $p \geq 1$.
8. A. Let X be any set consisting of more than one point. (i) Suppose $f, g : X \rightarrow X$ are such that $f \circ g = g \circ f$. Show that if f has a unique fixed point, then g has a fixed point. Can g have more than one fixed point? (Give a proof or counterexample.) (ii) Show that if we merely know that f has fixed points, it does not follow that g has any.

Reminder. $x \in X$ is a fixed point of $f : X \rightarrow X$ means $f(x) = x$.

9. A. Let $f : \mathbb{R} \rightarrow \mathbb{R} : f(x) = 1 + x + e^{-x}$ and let $W = [1, \infty)$. Show that f maps W into itself and satisfies $|f(x) - f(y)| < |x - y| \forall x \neq y$ in W , but does not have any fixed point in W . Why does the Contraction Mapping Theorem not apply?

Hint. If we haven't reached the Contraction Mapping Theorem by the time you are solving this problem, read the statement of the Theorem on page 1 of online notes for lecture 25.

0.4 Closed and open sets, continuity.

10. A. For a normed space $(V, \|\cdot\|)$ let \mathcal{U} denote the set of all those subsets of V that are open in $(V, \|\cdot\|)$. Prove the following properties of \mathcal{U} : (i) $\emptyset, V \in \mathcal{U}$, (ii) $U_1, \dots, U_n \in \mathcal{U} \Rightarrow \bigcap_{j=1}^n U_j \in \mathcal{U}$, (iii) $\forall j \in J, U_j \in \mathcal{U} \Rightarrow \bigcup_{j \in J} U_j \in \mathcal{U}$. These properties of \mathcal{U} make it what is called a topology; more on this in MA222 Metric Spaces.
11. A. Show that $f : (V, \|\cdot\|_V) \rightarrow (W, \|\cdot\|_W)$ is $(\|\cdot\|_V, \|\cdot\|_W)$ -continuous iff for every subset A closed in $(W, \|\cdot\|_W)$, $f^{-1}(A)$ is closed in $(V, \|\cdot\|_V)$.
12. B. Let $(V, \|\cdot\|)$ be a normed vector space over \mathbb{R} .

(i) Let $E \subset V$ be a non-empty subset of V . For each $x \in V$ define

$$d(x, E) = \inf_{y \in E} \|x - y\|.$$

Show that the map $d(\cdot, E) : V \rightarrow \mathbb{R}$ is continuous.

(ii) Let \mathcal{E} be the set of non-empty closed bounded sets in V . If $K, L \in \mathcal{E}$, define

$$\rho(K, L) = \max(\sup_{k \in K} d(k, L), \sup_{l \in L} d(l, K))$$

Prove that the function $\rho : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}$ is well defined and satisfies the following properties:

- (a) $\rho(L, K) = 0$ iff $L = K$;
 (b) For any $L, K \in \mathcal{E}$, $\rho(L, K) = \rho(K, L)$;
 (c) For any $L, K, M \in \mathcal{E}$, $\rho(L, K) \leq \rho(L, M) + \rho(M, K)$.

Hint. A subset of V is called bounded if it is contained in an open ball of a finite radius. For part (c) use the following inequality: for any $a, b, c, d \in \mathbb{R}$,

$$\max(a + b, c + d) \leq \max(a, c) + \max(b, d).$$

If in doubt, prove it using $2 \max(x, y) = x + y + |x - y|$.

Remark. A pair (X, d) where X is a set and $d : X \times X \rightarrow \mathbb{R}$ satisfies properties (a), (b), (c) listed above is called a **metric space**. In this exercise you proved that the set of closed bounded non-empty subsets of a normed vector space equipped with function ρ is a metric space. ρ is called the Hausdorff metric. It is widely used to compare closed bounded subsets of metric spaces.

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