NB. THESE ARE JUST SKELETON SOLUTIONS, **USE WISELY***!*

Questions for credit: 8 (5 points), 3 (7 points), 4 (7 points) and 7 (6 points)

0.1 Uniform convergence.

1. (Q.1)

   (a) Both series converge uniformly by the Weierstrass M-test: for $x \in [-M/2, M/2]$,

   $$\sum_{n=M}^{\infty} \frac{1}{(n \pm x)^2} \leq \sum_{n=M}^{\infty} \frac{1}{(n - M/2)^2} < \infty.$$

   Therefore, the uniform convergence of the series $f_M(x) = \sum_{n=M}^{\infty} \frac{1}{(n+x)^2} + \sum_{n=M}^{\infty} \frac{1}{(n-x)^2}$ on $[-M/2, M/2]$ follows from the M-test with $M_n = \frac{2}{(n-M/2)^2}$. Each of functions $h_n(x) := \frac{1}{(n-x)^2} + \frac{1}{(n+x)^2}$, $n = M, M+1, \ldots$ is continuous on [-M/2,M/2], hence the limit f_M is also continuous as the convergence is uniform. h_n's are continuously differentiable on $(-M/2, M/2)$, the series $\sum_{n=M}^{\infty} h_n'$ converges uniformly by the M-test with $M_n = \frac{4}{(n-M/2)^3}$ and the series $\sum_{n=M}^{\infty} h_n'$ converges. Therefore, f_M is differentiable on $(-M/2, M/2)$ and the derivative can be calculated by term-wise differentiation, $f_M' = \sum_{n=M}^{\infty} h_n'$.

   (b) We need to show that F is defined, continuous and differentiable on $(k, k+1)$ for every $k \in \mathbb{Z}$. Let $x \in (k, k+1)$. Suppose $k \geq 0$. Then

   $$F(x) = \sum_{p=-2(k+1)}^{2(k+1)} \frac{1}{(x-p)^2} + \sum_{p=2k+3}^{\infty} \frac{1}{(x-p)^2} + \sum_{p=2k+3}^{\infty} \frac{1}{(x+p)^2}.$$

   The finite sum in the right hand side of the above is continuous, differentiable for any $x \in (k, k+1)$. The sum of two infinite sums in the right hand side is continuous differentiable on $(-k-3/2, k+3/2)$ by Question 2 with $M = 2k+3$. As $(k, k+1) \subset (-k-3/2, k+3/2)$, we are done. The $k < 0$ case can be treated in exactly the same way. Periodicity: for any $N \in \mathbb{Z}$, $x \in \mathbb{R} \setminus \mathbb{Z}$,

   $$\begin{aligned} F(x+1) &= \lim_{N_1 \to -\infty} \sum_{n=N_1}^{N} \frac{1}{(x+1-n)^2} + \lim_{N_2 \to \infty} \sum_{n=N+1}^{N_2} \frac{1}{(x+1-n)^2} \\ &= \lim_{N_1 \to -\infty} \sum_{n=N_1-1}^{N-1} \frac{1}{(x-n)^2} + \lim_{N_2 \to \infty} \sum_{n=N}^{N_2-1} \frac{1}{(x-n)^2} \\ &= \lim_{N_1 \to -\infty} \sum_{n=N_1}^{N-1} \frac{1}{(x-n)^2} + \lim_{N_2 \to \infty} \sum_{n=N}^{N_2} \frac{1}{(x-n)^2} \\ &= \sum_{n=-\infty}^{N-1} \frac{1}{(x-n)^2} + \sum_{n=N}^{\infty} \frac{1}{(x-n)^2} = F(x). \end{aligned}$$

   All of the above operations are justified due to the pointwise convergence of F on $\mathbb{R} \setminus \mathbb{Z}$.

(c) (i) If g is continuous on $\mathbb{R}$, it is also continuous on $[0,1]$. Therefore, it is bounded on $[0,1]$ (Analysis II.) Then there exists $B \geq 0$: $|g(x)| \leq B$ for any $x \in [0,1]$. Any $y \in \mathbb{R}$ can be represented in the form $y = I + r$, where I is an integer and $r \in [0,1)$. Therefore, $|g(y)| = |g(I+r)| = |g(r)| \leq B$, where the second equality used the periodicity of g. (ii) f is bounded on $\mathbb{R}$. Therefore, there exists $B \geq 0$ such that for any $x \in \mathbb{R}$, $|f(x)| \leq B$. Using the functional relation,

$$|f(x)| \leq \frac{1}{4}\left(|f(x/2)| + |f(x/2+1/2)|\right) \leq \frac{1}{2}B,$$

which is true for any $x \in \mathbb{R}$. Repeating the above step n times we find that for any $x \in \mathbb{R}$, $|f(x)| \leq 2^{-n}B \to 0$ as $n \to \infty$. So $f \equiv 0$.

(d) (i) Apply L'Hopital's rule four times:

$$\lim_{x\to 0}(\pi^2 cosec^2(\pi x) - 1/x^2) = \pi^2 \lim_{x\to 0} \frac{x^2 - \sin^2(x)}{x^2 \sin^2(x)}$$

$$= \pi^2 \lim_{x\to 0} \frac{2x - \sin(2x)}{2x\sin^2(x) + x^2\sin(2x)} = \pi^2 \lim_{x\to 0} \frac{2 - 2\cos(2x)}{2\sin^2(x) + 4x\sin(2x) + 2x^2\cos(2x)}$$

$$= \pi^2 \lim_{x\to 0} \frac{4\sin(2x)}{6\sin(2x) + 12x\cos(2x) - 4x^2\sin(2x)}$$

$$= \pi^2 \lim_{x\to 0} \frac{8\cos(2x)}{12\cos(2x) + 12\cos(2x)} = \frac{\pi^2}{3}.$$

(ii) Function g is continuous on $\mathbb{R}\setminus\mathbb{Z}$. It is also continuous at $x=0$: note that f_1 is continuous on $[-1/2, 1/2]$. Therefore,

$$\lim_{x\to 0} g(x) = f_1(0) - \lim_{x\to 0}(\pi^2 cosec^2(\pi x) - 1/x^2) = f_1(0) - \pi^2/3 = g(0).$$

As F, $cosec^2(\pi\cdot)$ are both periodic with period 1 on $\mathbb{R}\setminus\mathbb{Z}$, g is continuous at every $x = n$, where $n \in \mathbb{N}$:

$$\lim_{x\to n} g(x) = \lim_{x\to 0} g(x+n) = \lim_{x\to 0} g(x) = g(0) = g(n).$$

So g is continuous on $\mathbb{R}$. It is periodic on $\mathbb{R}\setminus\mathbb{Z}$ with period 1 and $g(n) = g(0)$ for any $n \in \mathbb{Z}$. Therefore, g is periodic on $\mathbb{R}$ with period 1.

(e) (i) As proved in part (d), function g is continuous and periodic. Therefore, it is bounded by part (c). Consider the restriction of $g = F - \pi^2 cosec^2(\pi\cdot)$ to $\mathbb{R}\setminus\mathbb{Z}$. Then

$$\frac{1}{4}(F(x/2) + F((x+1)/2)) = \frac{1}{4}\left(\sum_{n\in\mathbb{Z}} \frac{1}{(x/2-n)^2} + \sum_{n\in\mathbb{Z}} \frac{1}{(x/2+1/2-n)^2}\right)$$

$$= \left(\sum_{n\in\mathbb{Z}} \frac{1}{(x-2n)^2} + \sum_{n\in\mathbb{Z}} \frac{1}{(x-2n+1)^2}\right)$$

$$= \left(\sum_{k\in 2\mathbb{Z}} + \sum_{k\in 2Z+1}\right) \frac{1}{(x-k)^2} = F(x).$$

All above steps are justified as the series for $F(x)$ converge absolutely for $x \in \mathbb{R}\setminus\mathbb{Z}$. Also,

$$\frac{1}{4}(cosec^2(\pi x/2) + cosec^2(\pi(x+1)/2)) = \frac{1}{4}(cosec^2(\pi x/2) + sec^2(\pi x/2))$$

$$= \frac{1}{4\sin^2(\pi x/2)\cos^2(\pi x/2)} = cosec^2(\pi x/2).$$

Conclusion: both F and $cosec^2(\pi\cdot)$ satisfy the equation of part (c) on $\mathbb{R}\setminus\mathbb{Z}$. As the equation is linear, g also satisfies it. As g is continuous at $x = n$, $n \in \mathbb{Z}$, we can take the limit of $x \to n$, to prove that the equation of part (c) is satisfied by g for any $x \in \mathbb{R}$. As g is bounded, we conclude that $g \equiv 0$, which means in particular that for any $x \in \mathbb{R}$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(x-n)^2} = \pi^2 cosec^2(\pi x).$$

(ii) For x close to zero, we can rewrite the above as

$$2\sum_{n=1} \frac{1}{(x-n)^2} = \pi^2 cosec^2(\pi x) - \frac{1}{x^2}.$$

The left hand side is $2f_1$, so it is continuous on $[-1/2, 1/2]$ and the series converges uniformly. Taking the limit $x \to 0$ term-wise is therefore justified and we get

$$\sum_{n=1} \frac{1}{(n)^2} = \pi^2/6.$$

0.2 Norms.

2. (Q.2) (i) $||\cdot||_B$ is not a norm, as it doesn't separate points: for any non-zero constant function $f \equiv c \neq 0$, $||f||_B = ||f'||_B = 0$, yet $f \neq 0$. Functions (1),(3),(4) do define norms on $C^1[0,1]$: (1) Separation of points: $||f||_A = 0 \Rightarrow ||f||_\infty = 0 \Rightarrow f = 0$ as $||\cdot||_\infty$ is a norm; $||f||_C = 0 \Rightarrow ||f||_\infty = 0 \Rightarrow f = 0$ as $||\cdot||_\infty$ is a norm; $||f||_D = 0 \Rightarrow f(0) = 0, f' \equiv 0$. As $f' \in C[0,1]$, the mean value theorem gives that for any $x \in [0,1]$ $f(x) = f(0) = 0$. (2) The absolute homogeneity is obvious for all cases. (3) Triangle inequality: all three candidates have the form $||f|| = ||L_1(f)||' + ||L_2(f)||''$, where $||\cdot||'$ and $||\cdot||''$ are norms and L_1, L_2 are linear operations (Such as differentiation or evaluation of f at a specific point). Then

$$||f+g|| = ||L_1(f+g)||' + ||L_2(f+g)||'' = ||L_1(f)+L_1(g)||' + ||L_2(f)+L_2(g)||''$$
$$\leq ||L_1(f)||' + ||L_1(g)||' + ||L_2(f)||'' + ||L_2(g)||'' = ||f|| + ||g||.$$

(ii) We claim that there are four equivalence classes: $(||\cdot||_\infty, ||\cdot||_A)$; $(||\cdot||_1)$; $(||\cdot||_C)$, $(||\cdot||_D)$. Let's check it:

- $||\cdot||_\infty \sim ||\cdot||_A$: for any $f \in C^1([0,1])$,

$$||f||_\infty \leq ||f||_A = ||f||_\infty + \int_0^1 |f| \leq ||f||_\infty + ||f||_\infty(1-0) = 2||f||_\infty.$$

  Therefore, $||\cdot||_\infty \sim ||\cdot||_A$.

- $||\cdot||_C \nsim ||\cdot||_D$: Let $f_k \mapsto x^k$ be a sequence of $C^1([0,1])$-functions. Then

$$||f_k||_D = ||f_k'||_1 = k\int_0^1 x^{k-1}dx = 1;$$

$$||f_k||_C = 1 + ||f'||_\infty = 1 + k \to \infty \text{ for } k \to \infty.$$

  Therefore, there is no $L > 0$ such that for any $g \in C^1([0,1])$, $||g||_C \leq L||g||_D$ and the norms are not equivalent.

- Now, $||\cdot||_\infty \nsim ||\cdot||_1$. Warning: we cannot simply refer to Prop. 38 (Lecture 20) to justify this result, as we are working in a different vector space. So, let $f_n(x) = \frac{1}{1+nx}, n = 1, 2, \ldots$ be a sequence of C^1-functions. $||f_n||_\infty = 1$ for all n's, but $||f_n||_1 = \int_0^1 \frac{1}{1+nx} dx = \frac{\log(1+n)}{n} \to 0$ as $n \to \infty$. Therefore, there exists no $K > 0$ such that for any $f \in C^1[0, 1]$, $K||f||_\infty \le ||f||_1$. The non-equivalence is proved. Therefore, $||\cdot||_A \nsim ||\cdot||_1$ as A- and ∞-norms belong to the same equivalence class.
- Similarly, $||\cdot||_\infty \nsim ||\cdot||_C$ and $||\cdot||_1 \nsim ||\cdot||_C$: consider $(f_n = sin^2(2\pi n\cdot))_{n\ge1} \subset C^1[0, 1]$. Then, $||f_n||_\infty = 1$, $||f_n||_1 = 1/2$. But $||f_n'||_\infty = 2\pi n \to \infty$ as $n \to \infty$. Therefore, there exist no $K_1 > 0, K_2 > 0$ such that $||f||_C \le K_1||f||_\infty$, $||f||_C \le K_2||f||_1$.
- Also, $||\cdot||_\infty \nsim ||\cdot||_D$ and $||\cdot||_1 \nsim ||\cdot||_D$: consider $(f_n = sin^2(2\pi n\cdot))_{n\ge1} \subset C^1[0, 1]$. Then, $||f_n||_\infty = 1$, $||f_n||_1 = 1/2$. But

  $$||f_n||_D = ||f_n'||_1 = \pi n \int_0^1 |\sin(4\pi nx)|dx = 2n \to \infty$$

  as $n \to \infty$. Therefore, there exist no $K_1 > 0, K_2 > 0$ such that $||f||_D \le K_1||f||_\infty$, $||f||_D \le K_2||f||_1$.

The claim is proved.

3. (Q.3) (a) Let $\lambda = 1/p$. Then $1 - \lambda = 1/q$. $\log(1/x)$ is convex on $(0, \infty)$. Let $x_1 = x^p$, $x_2 = y^q$. The convexity condition reads

$$\frac{1}{p}\log(1/x^p) + \frac{1}{q}\log(1/y^q) \ge \log(1/(x^p/p + y^q/q)).$$

Exponentiating both sides of the above and using that the exponential function is increasing, we find

$$1/(xy) \ge (x^p/p + x^q/q)^{-1},$$

which equivalent to (5) as $x, y > 0$. For $x = 0$ or $y = 0$ the inequality (5) remains valid, as at these special points both sides of (5) turn to zero.

(b) In virtue of (5), at any point $x \in [a, b]$, $|f(x)||g(x)| \le |f(x)|^p/p + |g(x)|^q/q$. In both sides of the inequality we have continuous functions. Using the theorem on integral bounds, we get:

$$\int_a^b |fg| \le \int_a^b |f|^p/p + \int_a^b |g|^q/q,$$

which holds for any $f, g \in C[a, b]$. If $F, G \in C[a, b]$: $\int_a^b |F|^p = \int_a^b |G|^q = 1$,

$$\int_a^b |FG| \le 1/p + 1/q = 1.$$

(c) Assume that f, g are not identically zero on $[a, b]$. Then $\int_a^b |f|^p > 0$, $\int_a^b |g|^q > 0$. Let $F = f/(\int_a^b |f|^p)^{1/p}$, $G = g/(\int_a^b |g|^q)^{1/q}$. Clearly, $\int_a^b |F| = 1$, $\int_a^b |G| = 1$. As we proved in part (b), $\int_a^b |FG| \le 1$. Re-writing this in terms of f, g again we get the desired inequality:

$$\int_a^b |fg| \le \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^q\right)^{1/q}.$$

We proved the inequality for any f, g which are not identically zero on $[a,b]$. It is easy to check that it is also valid for $f \equiv 0$ or $g \equiv 0$. For $p = q = 2$,

$$\int_a^b fg \le \int_a^b |fg| \le \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$

4. (Q.4) (a) Assume that f is not identically zero. Choose $\alpha = p - 1$. Then $q^{-1} = 1 - p^{-1}$ and

$$v \int_a^b |f|^p \le A \left(\int_a^b |f|^{q(p-1)}\right)^{1-p^{-1}}.$$

But $q(p-1) = p$. Dividing both sides with $\left(\int_a^b |f|^{q(p-1)}\right)^{1-p^{-1}} > 0$ we find

$$\left(\int_a^b |f|^p\right)^{1/p} \le A,$$

as required. A quick inspection shows that the above inequality is valid for $f \equiv 0$, thus completing the proof.

(b) Applying Holder's inequality we find that for any $g \in C[a,b]$,

$$\int_a^b |(f+h)g| \le A \left(\int_a^b |g|^q\right)^{1/q},$$

where $A = \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |h|^{1/p}\right)^{1/p}$. By the inequality proved in part (a),

$$\left(\int_a^b |f+h|^p\right)^{1/p} \le A = \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |h|^{1/p}\right)^{1/p}.$$

Minkowski's inequality is proved.

(c) Let $||f||_p = \left(\int_a^b |f|^p\right)^{1/p}$. Minkowski's inequality means that the function $||\cdot||_p : C[a,b] \to \mathbb{R}$ satisfies the triangle inequality. The absolute homogeneity is a doddle to prove. The separation of points: if $||f||_p = 0$, then $|f|^p \equiv 0$, as $|f|^p$ is continuous. (See the notes for the proof of $p = 1$ case.) Therefore $f \equiv 0$ and we conclude that $||\cdot||_p$ is indeed a norm for any $p \ge 1$.

0.3 Completeness.

5. (Q.5) In Question 2 we identified four equivalence classes of norms. Using the statement quoted in the hint, it is enough to check completeness for just one representative of each class:

(1) The space $(C^1[0,1], ||\cdot||_\infty)$ is not complete: the limit of a uniformly Cauchy sequence of C^1-functions has to be continuous, but it doesn't have to be differentiable. An example was given in the lectures: $f_n(x) = \sqrt{(x-1/2)^2 + 1/n}, n = 1, 2, \ldots$. (Full details omitted.) By equivalence, $(C^1[0,1], ||\cdot||_A)$ is not complete either.

(2) The space $(C^1[0,1], ||\cdot||_1)$ is not complete. The sequence $(f_n(x) = \tanh(n(x-1/2))_{n\geq 1} \subset C^1[0,1]$ is Cauchy w. r. t. $||\cdot||_1$ (sketch a picture):

$$||f_{n+k} - f_n||_1 = \int_0^1 |\tanh((n+k)(x-1/2)) - \tanh(n(x-1/2))|$$

$$= \left(\int_0^{1/2-1/\sqrt{n}} + \int_{1/2-1/\sqrt{n}}^{1/2+1/\sqrt{n}} + \int_{1/2+1/\sqrt{n}}^1\right) |\tanh((n+k)(x-1/2)) - \tanh(n(x-1/2))|$$

$$:= I_1 + I_2 + I_3.$$

But $I_2 \leq 4/\sqrt{n} \to 0$ as $n \to \infty$;

$$I_3 = \int_{1/2+1/\sqrt{n}}^1 |\tanh((n+k)(x-1/2)) - \tanh(n(x-1/2))|$$

$$\stackrel{MVT}{=} \int_{1/2+1/\sqrt{n}}^1 \frac{1}{\cosh^2 \xi(n,k,x)} |k(x-1/2)|,$$

where $\xi(n,k,x) \in (n(x-1/2), (n+k)(x-1/2))$. Therefore, $\xi(n,k,t) \geq \sqrt{n} \to \infty$ as $n \to \infty$. So, $I_3 \to 0$ as $n \to \infty$. Similarly, $I_1 \to 0$ as $n \to \infty$. Therefore, (f_n) is $||\cdot||_1$ Cauchy. However, the sequence (f_n) does not have a limit in $C^1[0,1]$. In fact, an argument similar to the one used in the lectures, shows that (f_n) cannot even converge to a continuous function on $[0,1]$.
(3) The space $(C^1[0,1], ||\cdot||_C)$ is Banach. Really, any sequence $(f_n)_{n\geq 1} \subset C^1[0,1]$ which is Cauchy with respect to $||\cdot||_D$ must have the following properties: (i) (f_n) is uniformly Cauchy; (ii) (f_n') is uniformly Cauchy. Therefore, we are dealing with a pointwise convergent series of C^1-functions such that the series of derivatives converges uniformly. The limit must be C^1 by Theorem 25. This shows that any Cauchy sequence in $(C^1[0,1], ||\cdot||_C)$ converges, which proves completeness.
(4) The space $(C^1[0,1], ||\cdot||_D)$ is not Banach. Let $g_n \in C^1[0,1]$ be the following sequence: $g_n(0) = 0$ and $g_n'(x) = \tanh(n(x-1/2))$. Notice that $||g_n||_D = ||g_n'||_1$ and we know from part (2) that the sequence of g_n is Cauchy with respect to $||\cdot||_D$. We also know from part (2) that the sequence (g_n') does not converge to any continuous function on $[0,1]$. So, the sequence of (g_n) does not converge to any differentiable function: let $L \in C^1[0,1]$. Then there is $l \in C[0,1] : L(x) = L(0) + \int_0^x l(t)dt$. Therefore, $||g_n - L||_D = |L(0)| + ||g_n' - l||_1$ - doesn't converge to zero as $n \to \infty$.

6. (Q.6) If $(a_n)_{n\geq 1}, (b_n)_{\geq 1} \in s_{00}$, there exists $N \in \mathbb{N}$: for any $n > N$, $a_n = b_n = 0$. Therefore $c_n := a_n + b_n = 0$, $d_n := \lambda a_n = 0$ for $n > N$ and any $\lambda \in \mathbb{R}$. Therefore $(c_n)_{n\geq 1}, (d_n)_{n\geq 1} \in s_{00}$. Therefore, s_{00} is closed under component-wise addition and multiplication by scalars. One can also check that these operations satisfy all linear space axioms, implying that s_{00} is a linear space over $\mathbb{R}$.

   Consider the following sequence of sequences: $(\frac{1}{2^n}\mathbb{I}\{n \leq N\})_{n\geq 1} \subset s_{00}$, $N = 1, 2, \ldots$. Here $\mathbb{I}\{n \leq N\} = 1$ if $n \leq N$ and is zero otherwise. This sequence is Cauchy:

$$||(a)_{n\geq 1}^{N+M} - (a)_{n\geq 1}^N||_1 = \sum_{k=N+1}^{N+M} 2^{-k} \leq \sum_{k=N+1}^{\infty} 2^{-k} \to 0,\ N \to \infty.$$

However, there is no sequence in s_{00} to which $(a_n)_{n\geq 1}^N$ converges: consider an arbitrary $(l_n)_{n\geq 1} \in s_{00}$. Then there exists $C \in N$ such that $l_n = 0$ for any $n \geq C$. Therefore for any $N > C$,

$$||(a_n)_{n\geq 1}^N - (l_n)_{\geq 1}||_1 \geq 2^{-C} \nrightarrow 0 \text{ as } N \to \infty.$$

7. (Q.7) Consider the following sequence of piece-wise linear functions (already used in the lectures to show that $(C[a,b]), ||\cdot||_1$ is not complete): $(f_n)_{n\geq 1} \subset C[a,b]$, $f_n(x) = -1$ for $x \leq (a+b)/2 - (b-a)/(2n)$, $f_n(x) = 1$ for $x \geq (a+b)/2 + (b-a)/(2n)$,

$$f_n(x) = 2n\frac{(x-(a+b)/2)}{(b-a)}$$

for any other $x \in [a,b]$. For any $m, n \in \mathbb{N}$,

$$||f_{n+m} - f_n||_p \leq \left(\int_{(a+b)/2-(a-b)/(2n)}^{(a+b)/2+(a-b)/(2n)} 2^p dx\right)^{1/p} \leq 2((a-b)/n)^{1/p} \to 0 \text{ as } n \to \infty.$$

Therefore, the sequence of functions $(f_n)_{n\geq 1}$ is Cauchy with respect to $||\cdot||_p$.

The fact that the sequence cannot converge to a continuous function can be proved either by repeating the argument given in the lectures or as follows: let $l \in C[a,b]$. It is continuous at the midpoint $m = (a+b)/2$. Then for any $\epsilon > 0$, there is $\delta > 0$: for any $x \in (m-\delta, m+\delta)$, $|f(x) - f(m)| < \epsilon$. Then for any n: $(b-2)/(2n) < \delta/2$,

$$||f_n - l||_p$$
$$\geq \frac{\delta}{2}\left(\min(\inf_{x\in(f(m)-\epsilon, f(m)+\epsilon)} |1+x|^p, \inf_{x\in(f(m)-\epsilon, f(m)+\epsilon)} |1-x|^p)\right)^{1/p} > 0$$

for $\epsilon < 1/2$, as no interval of length $2\epsilon < 1$ can contain both $+1$ and -1. Notice that the lower bound we derived does not depend on n. Therefore, $||f_n - l|| \nrightarrow 0$ as $n \to \infty$. We found a Cauchy sequence, which does not converge in $(C[a,b], ||\cdot||_p)$. Therefore, L_p is not complete for any $p \geq 1$.

8. (Q.8) (i) Let $x \in X : f(x) = x$ be the unique fixed point of f. Applying $f \circ g = g \circ f$ to x we get $f(g(x)) = g(f(x)) = g(x)$. Therefore, $g(x)$ is a fixed point of f. As the fixed point is unique, we must have $g(x) = x$, meaning that x is also a fixed point of g. Yes, g can have more than one fixed point: let $g = Id$, the identity map. Then every point of $x \in X$ is a fixed point of X $(g(x) = Id(x) = x)$ and $f \circ g = f \circ Id = f = Id \circ f = g \circ f$. (ii) Suppose $f : \mathbb{R} \to \mathbb{R}$ is the identity map and $g : x \in \mathbb{R} \mapsto 1 + |x|$. Then f leaves every point of $\mathbb{R}$ fixed, $f \circ g = g \circ f$, but g has no fixed points.

9. (Q.9) $x + e^{-x} \geq 0 \; \forall x \geq 1$ so f maps $U = [1, \infty)$ into itself. If $x \neq y$, by the Mean Value Theorem $|f(x) - f(y)| = |f'(\xi)|\,|x-y|$ for some ξ between x and y; now $|f'(\xi)| = |1 - e^{-\xi}| < 1$ for any $\xi \in [1, \infty)$ so $|f(x) - f(y)| < |x - y|$.
f has no fixed point since $f(x) = x \iff 1 + e^{-x} = 0$ which is impossible for $x \in \mathbb{R}$ and hence in U. The Contraction Mapping Theorem does not apply because there exists no contraction factor $K < 1$ such that $|f(x) - f(y)| < K|x-y| \; \forall x \neq y \in U$. Indeed given $K < 1$ take x, y and hence $\xi > \log\left(\frac{1}{1-K}\right)$ then $1 - e^{-\xi} > K$ so $|f(x) - f(y)| = |f'(\xi)|\,|x-y| > K|x-y|$.

0.4 Closed and open sets, continuity.

10. (Q.10) $U \in \mathcal{U}$ if, $\forall x \in U$, $\exists \varepsilon > 0$ s.t. $B(x,\varepsilon) \subset U$. (i) is easy, because $\nexists x \in \emptyset$ and any $B(x,\varepsilon) \subset V$.
(ii) For $x \in \bigcap_{j=1}^{n} U_j$, $\forall j \in \{1,\ldots,n\} \exists \varepsilon_j > 0$ with $B(x,\varepsilon_j) \subset U_j$ so, for $\varepsilon := \min\{\varepsilon_j : 1 \leq j \leq n\}$ we have $B(x,\varepsilon) \subset \bigcap_{j=1}^{n} U_j$.
(iii) For $x \in \bigcup_{j\in J} U_j$, $\exists j \in J$ s.t. $x \in U_j$ and $\exists \varepsilon$ s.t. $B(x,\varepsilon) \subset U_j \subset \bigcup_{j\in J} U_j$ as required.

11. (Q.11) Follows from $f^{-1}(W \backslash A) = V \backslash f^{-1}(A)$ for any $A \subset W$ and $W \backslash A$ is open iff A is closed.

12. (Q.12) (i) For any $x, z \in V$,

$$d(x+z,E) = \inf_{y\in E} ||x+z-y|| \leq \inf_{y\in E}(||x-y|| + ||z||) = d(x,E) + ||z||,$$

$$d(x,E) = \inf_{y\in E} ||x+z-y-z|| \leq \inf_{y\in E}(||x+z-y|| + ||z||) = d(x+z,E) + ||z||,$$

So,

$$|d(x+z,E) - d(x,E)| \leq ||z||.$$

Therefore, the map $d(\cdot, E)$ is continuous at every $x \in E$ by the second definition of continuity (Definition $14'$ of the notes) with $\delta = \epsilon$.

(ii) (a) $\Leftarrow$ (easy direction): Suppose $K = L$. For any $k \in L$, $d(k,L) = \inf_{y\in L} d(k,L) = 0$, (the infimum is reached at $y = k$). Therefore,

$$\rho(L,L) = \max(\sup_{k\in L} d(k,L), \sup_{l\in L} d(l,L)) = \sup_{k\in L} d(k,L) = \sup_{k\in L}(0) = 0.$$

$\Rightarrow$ (a harder direction): Suppose $\rho(K,L) = 0$. Then $\sup_{k\in K} d(k,L) = 0$ and $\sup_{l\in L} d(l,K) = 0$. Consider the first condition. It implies that for any $k \in K$, $\inf_{l\in L} ||k-l\,|| = 0$. Consider any null sequence of positive numbers $(\delta_n)_{n\geq 1}$. As $\inf_{l\in L} ||k-l\,|| = 0$, for any $\delta_n > 0$ there is $l_n \in L : ||k-l_n|| \leq \delta_k$. The sequence $(\delta_k)_{k\geq 1}$ converges to k. **As L is closed**, this implies that $k \in L$. (Proposition 45 of the notes). We conclude that $K \subset L$. Analysing the second condition $\sup_{l\in L} d(l,K) = 0$ in exactly the same way we arrive at $L \subset K$. Therefore $K = L$.

(b) This is obvious as $\max(x,y) = \max(y,x)$.

(c) For any $m \in M$, $k \in K$

$$d(k,L) := \inf_{l\in L} ||k-l\,|| \leq \inf_{l\in L}(||k-m)|| + ||m-l\,||).$$

Taking $\inf_{m\in M}$ of both sides of the above, we get

$$d(k,L) \leq d(k,M) + \inf_{m\in M} d(m,L) \leq d(k,M) + \sup_{m\in M} d(m,L).$$

Similarly, for any $l \in L$,

$$d(l,K) \leq d(l,M) + \sup_{m\in M} d(m,K).$$

Substituting the last two inequalities in the expression for ρ we get

$$\rho(K,L) \leq max(\sup_{k\in K} d(k,M) + \sup_{m\in M} d(m,L), \sup_{l\in L} d(l,M) + \sup_{m\in M} d(m,K))$$

$$\leq max(\sup_{k\in K} d(k,M), \sup_{m\in M} d(m,K)) + max(\sup_{m\in M} d(m,L), \sup_{l\in L} d(l,M))$$
$$= \rho(K,M) + \rho(M,L).$$

At the last step we used the magic formula from the hint.

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