MA134 Geometry and Motion

Examples Sheet 10 2019

These examples are not for credit but they are part of the course so you should do them.

---

1. Let

$$f(x,y) = 3x^2 - y \quad \text{and} \quad \mathbf{F} = (y-2x)\mathbf{i} + (3x+2y)\mathbf{j}$$

Compute the line integrals

$$\oint_{\mathcal{C}} f\,ds \quad \text{and} \quad \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

where $\mathcal{C}$ is a circle of radius 2 centred on the origin and $\mathcal{C}$ is traversed in a counterclockwise direction. Show explicitly that the if $\mathcal{C}$ is traversed in the clockwise direction instead that the first integral does not change while the second one changes sign.

2. Compute the line integral

$$\int_{\mathcal{C}} xyz\,ds$$

where $\mathcal{C}$ is parametrised by $\mathbf{r}(t) = (2\sin t, t, -2\cos t)$, $0 \le t \le \pi$.

3. Compute the line integral

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

for $\mathbf{F} = (x+y, y^2-x)$ and $\mathcal{C}$ is the curve which begins at $(-1,0)$, proceeds along the x-axis to $(1,0)$ and returns to $(-1,0)$ by the upper part of the unit circle.

4. Compute the line integral

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

for $\mathbf{F} = (\frac{1}{xy^2}, \frac{1}{x^2y})$ and $\mathcal{C}$ is the curve given by $\mathbf{r}(t) = (\sqrt{t}, \sqrt{1+t})$, for $t \in [1,4]$.

5. An important vector field is the inverse square force

$$\mathbf{F}(\mathbf{r}) = -k\frac{\mathbf{r}}{r^3},\ \mathbf{r} \in \mathbb{R}^3 \setminus \{\mathbf{0}\},$$

where $k > 0$ is a constant, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $r = \|\mathbf{r}\|$. Gravitational and Coulomb forces generated from point source (mass or charge) at $\mathbf{r} = \mathbf{0}$ are of this form. The vector field is conservative with potential $V(\mathbf{r}) = -\frac{k}{r}$.

(a) Verify that $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$. Show that $\|\mathbf{F}(\mathbf{r})\| = \frac{k}{r^2}$, (this is the inverse square law).

(b) Verify by direct integration that

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = 0$$

For $\mathcal{C}$ any circle centred on the origin.

(c) Find

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ starts at $(1, 0, 0)$ and ends at $(2, 2, 0)$. Do this is two ways. First by using the potential and second by line integration. Do not choose $\mathcal{C}$ to be a straight line connecting $(1, 0, 0)$ and $(2, 2, 0)$. Instead consider a curve consisting of a arc of circle about the origin and a radial line. (Note, you can connect any two points with a curve consisting of an arc of a circle at constant r and a radial line.)

6. Show that the vector field

$$\mathbf{F} = \sin x \cos y \cos z\, \mathbf{i} + \cos x \sin y \cos z\, \mathbf{j} + \cos x \cos y \sin z\, \mathbf{k}$$

is a conservative vector field.

7. Show that the vector field

$$\mathbf{F} = (1 + \sin x \cos z) \cos y\, \mathbf{i} + \cos x \sin y \cos z\, \mathbf{j} + \cos x \cos y \sin z\, \mathbf{k}$$

is not a conservative vector field.

8. The velocity field for a point vortex in two dimensions is

$$\mathbf{v} = \frac{\alpha}{x^2 + y^2}(-y, x)$$

where α is a parameter quantifying the strength of the vortex. Compute the circulation

$$\Gamma = \oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r}$$

for 2 cases: (i) $\mathcal{C}$ is a circle of radius R about the origin and (ii) $\mathcal{C}$ is a square with vertices $(a, -a)$, (a, a), $(-a, a)$, and $(-a, -a)$, so that $2a$ is the length of a side. In both cases $\mathcal{C}$ is traversed in a counterclockwise direction.