MA134 Geometry and Motion

EXAMPLES SHEET 2

Answers to Part B questions must be handed in to your supervisor via the pigeon loft by 2pm Thursday, 24th January, 2019 (week 3).

Part A. Easier and background questions to be done first. Not to be handed in for marking.

1. Consider a particle moving according to the parametrisation

$$\mathbf{r}(t) = (\frac{1}{2}t^2, \frac{4}{5}t^{5/2}, \frac{2}{3}t^3), \quad 0 \le t \le 4,$$

where t is time. Find the particle's velocity, speed, acceleration. Calculate the length of the path.

2. Suppose the plane curve $\mathcal{C}$ is given in terms of polar coordinates $(r(t), \theta(t))$, $t \in [a, b]$. Show that the length of $\mathcal{C}$ is given by:

$$\ell(C) = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2}\, dt.$$

3. Using the formula in the previous question, compute the length of the curve

$$(r, \theta) = (2\cos t, t), \quad t \in [0, \pi]$$

4. Verify the following differentiation formulas by writing them out in component form:

(a) $\dfrac{d}{dt}\big(f(t)\cdot \mathbf{s}(t)\big) = f'(t)\cdot \mathbf{s}(t) + f(t)\cdot \mathbf{s}'(t)$

(b) $\dfrac{d}{dt}\big(\mathbf{r}(t)\cdot \mathbf{s}(t)\big) = \mathbf{r}'(t)\cdot \mathbf{s}(t) + \mathbf{r}(t)\cdot \mathbf{s}'(t)$

(c) $\dfrac{d}{dt}\mathbf{r}(f(t)) = \mathbf{r}'(f(t))f'(t)$

5. Suppose $\mathbf{r}(t) \neq \mathbf{0}$, show that

$$\frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t)\cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}.$$

B. Questions for credit

6. Let $\mathbf{r}(t)$ be a parametrisation of curve $\mathcal{C}$ in $\mathbb{R}^3$ such that $\mathbf{r}(0) = (R, -R, R)$, where $R \in \mathbb{R}$. Suppose $\mathbf{r}(t) \neq \mathbf{0}$ and $\mathbf{r}(t)\cdot \mathbf{r}'(t) = 0$ for all points of $\mathcal{C}$. Show that *any* such $\mathcal{C}$ must lie on the surface of a sphere. Find the position of the sphere's centre and determine its radius.

7. Assume a particle moves along the path given by

$$\mathbf{r}(t) = (\cos^2 t, \cos t),\ t \in \mathbb{R}_+$$

where t corresponds to time. Sketch the path. What is the shape of the path? Find the points where the velocity is zero and find the acceleration at these points. Describe the particle's motion.

8. In two dimensions, mark a point on the outer rim of a wheel of radius R_0 rolling on the horizontal line $y = 0$ without slippage. Assume that the centre of the wheel is moving with velocity $\mathbf{v} = V_0\mathbf{i},\ V_0 > 0$ and that the marked point is at $(0, 0)$ at time $t = 0$.
(a) Find $\mathbf{r}(t)$, the position of the marked point at time t. The curve traced out by the marked point is called a *cycloid*.
(b) Compute the the velocity $\mathbf{v}(t)$, speed $c(t)$, and acceleration $\mathbf{a}(t)$ of the moving point.
(c) Sketch the path of the point $\mathbf{r}(t)$ (over some reasonable range of t). Identify all points at which $\mathbf{v}(t) = 0$ and mark them on your sketch.
(d) Compute the length of the path between two consecutive points where the path touches $y = 0$.

9. Compute the length of the segment of the cardioid

$$(r, \theta) = (2 - 2\cos t, t), \quad t \in [0, 2\pi]$$

lying in the first quadrant.

10. Reparametrise the curve

$$\mathbf{r}(t) = \left(\frac{1}{t^2+1} - \frac{1}{2}\right)\mathbf{i} + \frac{t}{t^2+1}\mathbf{j}, \quad t \geq 0$$

with respect to arc length measured from the point $(1/2, 0)$. Express the reparametrisation in its simplest form. What can you conclude about the curve?

11. An important class of curves on surfaces are *geodesics*. These are shortest paths (at least locally) between two points on the surface. In later modules you may study these in depth. The aim of the present question is to construct geodesics on the round cylinder. To achieve this, we will use the following two facts without proof:

(i) The shortest path between two points in the plane is a segment of the straight line passing through these points.
(ii) A round cylinder of radius R can be obtained from the strip $[0, 2\pi R] \times \mathbb{R} \subset \mathbb{R}^2$ by folding the strip and gluing all points $(0, z)$ and $(2\pi R, z)$, $z \in \mathbb{R}$. The process of folding/unfolding preserves the lengths of all curves on the cylinder.

The above two statements mean that after cutting the cylinder along any line parallel to its axis and lying it flat, all geodesics will locally be segments of straight lines. The equation defining our cylinder in $\mathbb{R}^3$ is $x^2 + y^2 = R^2$. All points on the cylinder can be uniquely parametrised as $(R\cos\theta, R\sin\theta, z)$, $\theta \in [0, 2\pi)$, $z \in \mathbb{R}$ (the so called cylindrical coordinates, which we will revisit later in the course). We are now ready to construct the shortest path between the points $P = (R, 0, 0)$ and $Q = (R\cos\phi, R\sin\phi, h)$ on the cylinder. Here $0 \leq \phi < \pi$, $h \geq 0$.

(a) Construct a bijective map U between the cylinder and the strip $[0, 2\pi R) \times \mathbb{R} \subset \mathbb{R}^2$ which corresponds to unfolding of the cylinder after cutting it along the line $x = R, y = 0$ in $\mathbb{R}^3$.

(b) Construct the shortest path connecting the points $U(P)$ and $U(Q)$ and parametrise the corresponding curve. Invert the map U to construct the parametrisation of the geodesic passing through the points P and Q on the cylinder.

(c) Identify the curve you you found with one of the curves in $\mathbb{R}^3$ discussed in the lectures. Also, consider the following special cases: $\phi = 0, h > 0$ and $\phi > 0, h = 0$, and identify the corresponding curves.

Hint. You may work in groups for this part of the homework. Seek guidance from your supervisor if necessary.

C. More exercises.
(They should not be handed in for marking.)

12. Let the curve $\mathcal{C}$ be the set of points satisfying the equations

$$x^2 = 2y \quad 3z = xy$$

Find the length of $\mathcal{C}$ from the origin to the point $(6, 18, 36)$. (You should first find a parametrisation for $\mathcal{C}$.)

13. Suppose you start at the point $(0, 0, 3)$ and move a distance of 5 units along the curve $\mathbf{r}(t) = (3\sin t, 4t, 3\cos t)$ in the positive t direction. Where are you now?

14. For the path and sketch in question 9, show some velocity vectors. What is the speed of the point when it touches the ground $y = 0$? When a car is *drifting* the wheels slip along the ground and the car moves sideways. The path of a point on the rim will be more like

$$\mathbf{r}(t) = \left(\alpha t - R\sin\frac{t}{R}\right)\mathbf{i} + \left(R - R\cos\frac{t}{R}\right)\mathbf{j} + \beta t\mathbf{k}, \quad t \in \mathbb{R}$$

where $0 \le \alpha < 1$, $0 \le \beta < 1$. Sketch this path for: $\alpha = 0$ and β small; α small and $\beta = 0$; α and β small.

15. Show that arc length of a curve is independent of parametrisation. You can restrict to considering regular parametrisations that give the same orientation to the curve. You need to show that any two such parametrisations give the same arc length. This is a good problem to discuss with your supervisor.