MA134 Geometry and Motion

Examples Sheet 3

Answers to Part B questions must be handed in to your supervisor via the pigeon loft by 2pm Thursday, 31st January 2019 (week 4).

Part A. Easier and background questions to be done first. Not to be handed in for marking.

1. Using the alternative formula for curvature, find the curvature of

$$(a)\ \ \mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j} \qquad\qquad (b)\ \ \mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + (1+t^2)\,\mathbf{k}$$

2. For the logarithmic spiral

$$\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j},\ \ t \in \mathbb{R},$$

compute the unit tangent and principle normal vectors at $t = 0$ and plot them on the curve. Using the alternative formula for curvature, compute the curvature for all $t \in \mathbb{R}$. What is the radius of curvature at $t = 0$? Plot the osculating circle at $t = 0$.

3. In a screencast from Week 2 Pr. Barkley shows how to reparametrise a helix with respect to arc length to obtain

$$\mathbf{r}_{\text{arc}}(s) = \left(\frac{4s}{5}, 3\sin\frac{s}{5}, 3\cos\frac{s}{5}\right),\ \ s \geq 0.$$

For this parametrisation, compute $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, ρ, and τ. There are many things to compute here, but each computation is straightforward since the parametrisation is in term of arc length.

B. Questions for credit

4. Consider the curve $\mathcal{C}$ parametrised by

$$\mathbf{r}(t) = (t/2 + \frac{1}{2}\sin t, \frac{1}{2}\cos t, 2\cos(t/2)), \quad t \in \mathbb{R}$$

(i) Prove that the above is an arc length parametrisation of $\mathcal{C}$.

(ii) Calculate the unit tangent vector, principle normal vector, binormal vector, and curvature of $\mathcal{C}$ at $(\frac{\pi}{2}, -\frac{1}{2}, 0)$. Determine the osculating plane at this point.

5. Consider the ellipse satisfying the equation

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

(i) Calculate the curvature at every point of the ellipse. Find all points at which the curvature is either maximal or minimal. You will find that there are four such points.

(ii) Plot the ellipse and infer the principle normal vector $\mathbf{N}$ at all these points from the plot. (In this problem you are not asked to explicitly compute $\mathbf{N}$ at these points, so don't try.)

(iii) Give the locations of the centres of the osculating circles corresponding to the four points and plot the four osculating circles. (Plots may be hand drawn or produced by computer.)

6. Use Frenet-Serret formulae

$$\mathbf{T}' = \kappa\mathbf{N}, \mathbf{N}' = -\kappa\mathbf{T},$$

where $'$ denotes the derivative with respect to the natural parameter s, which we derived in the lectures, to describe all planar curves with constant curvature $\kappa > 0$.

Hint. Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ be a natural parametrisation of a constant curvature curve Γ. Calculate $\frac{d}{ds}(\mathbf{r}(s) + \frac{1}{\kappa}\mathbf{N}(s))$ remembering that κ is a constant. Use your answer and the fact that $||\mathbf{N}(s)|| = 1$ to determine the shape of Γ.

7. Describe all curves in $\mathbb{R}^n$ with constant zero curvature. **Hint.** Use natural parametrisation.

C. More exercises.
(They should not be handed in for marking.)

8. At what point does the curve $\mathbf{r}(t) = (t, e^t), t \in \mathbb{R}$, have maximum curvature? What is the limit of the curvature as $t \to \infty$?

9. For the curve

$$\mathbf{r}(t) = a(3t - t^3)\,\mathbf{i} + 3at^2\,\mathbf{j} + a(3t + t^3)\,\mathbf{k}$$

show that $\kappa(t) = \tau(t) = 1/3a(1 + t^2)^2$.

10. Describe all curves in $\mathbb{R}^3$ with constant curvature $\kappa > 0$ and constant torsion $\tau > 0$.