MA134 Geometry and Motion

Examples Sheet 4

Answers to Part B questions must be handed in to your supervisor via the pigeon loft by 2pm Thursday, 7th February 2019 (week 5).

Part A. Easier and background questions to be done first. Not to be handed in for marking.

1. Find all the first partial derivatives of the following functions of several variables

   (a) $f(x,y) = y^5 - 3xy$ (b) $f(x,t) = e^{-t}\cos \pi x$ (c) $g(x,y) = \dfrac{x-y}{x+y}$

2. Compute the gradient vector ∇f for the following:

   (a) $f(x,y) = xe^{xy}$ (b) $f(x,y) = \log\left(\frac{x}{x+y}\right)$

3. Use the Chain Rule to find dg/dt with $g(t) = f(x(t), y(t))$ for $f(x,y) = x^2+y^2+xy$; $x(t) = \sin t$, $y(t) = \cos t$

   Verify the result by first forming $g(t)$ and differentiating it directly.

4. Consider the function

$$f(x,y) = \frac{y^2}{x}.$$

   Compute $D_{\mathbf{u}}f(1,2)$ for $\mathbf{u} = (2/3, \sqrt{5}/3)$. Compute $D_{\mathbf{u}}f(1,2)$ in two ways, first using the gradient vector and second using explicitly the definition of $D_{\mathbf{u}}f$ as a limit.

5. Compute all the second derivatives $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial z^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial z}, \dfrac{\partial^2 f}{\partial x \partial z}$, for $f(x,y,z) = xe^y \sin \pi z$.

B. Questions for credit

6. Find all the first partial derivatives of the following functions of several variables

   (a) $f(y,t) = t^2 e^{\frac{y}{t}}, y \in \mathbb{R}, t > 0$; (b) $f(p,q) = qe^{\frac{1}{p}} + pe^{\frac{1}{q}}, p, q > 0$; (c) $h(x,y,z,t) = xyz^2\cot(yt), (x,y,z,t) \in \mathbb{R}^4$.

7. Compute the gradient vector ∇f for the following:

   (a) $f(x,y) = (x+y)e^{x+y}, (x,y) \in \mathbb{R}^2$ (b) $f(x,y,z) = \sin(xyz), (x,y,z) \in \mathbb{R}^3$
   (c) $f(\mathbf{r}) = G(\|\mathbf{r}\|^2)$, $\mathbf{r} \in \mathbb{R}^n$, $G : \mathbb{R} \to \mathbb{R}$ any differentiable function.

8. Use the Chain Rule or any other method to find dg/dt for
   $g(t) = G(x(t)^2 - y(t)^2)$; $x(t) = A\cosh(t)$, $y(t) = A\sinh(t)$
   Here A is a constant, $t \in \mathbb{R}$ and G is a differentiable function on $\mathbb{R}$.

9. Compute the directional derivative $D_{\mathbf{u}}f$ of

$$f(x, y, z) = xe^{2yz}$$

at the point (0,0,0) in the direction $\mathbf{u} = (2/3, -2/3, 1/3)$ using the gradient vector.

10. A function $f : \mathbb{R}^n \backslash \mathbf{0} \to \mathbb{R}$ is called homogeneous of degree D if for every $\lambda \in \mathbb{R}\backslash 0$ and $(x_1, \dots, x_n) \in \mathbb{R}^n \backslash \mathbf{0}$ we have the equality

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^D f(x_1, x_2, \dots, x_n). \qquad (*)$$

Show that any homogeneous differentiable function f of degree D satisfies

$$\sum_{k=1}^{n} x_k \frac{\partial f}{\partial x_k} = D \cdot f \ \forall (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \setminus \mathbf{0}.$$

Hint. Use the chain rule to calculate the λ-derivative of the definition (*) at $\lambda = 1$.

11. Show that any function $u(x, t)$ of the form

$$u(x, t) = f(x + ct) + g(x - ct)$$

where f and g are twice differentiable, satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Here c is a constant (e. g. the speed of light). What is the meaning of the solutions described by functions f and g?

C. More exercises.
(They should not be handed in for marking.)

12. Show that $u(x, y) = \ln \sqrt{x^2 + y^2}$ satisfies the Laplace equation in 2D

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ (x, y) \neq (0, 0)$$

13. Show that $u(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation in 3D

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, (x, y, z) \neq (0, 0, 0)$$

14. Let

$$u(x, t) = f(kx + \omega t)$$

where f is twice differentiable. Find the relationship between k, ω and a so that u satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

15. Assume u and v are differentiable functions of x and y and that a and b are constants. Show that taking the gradient satisfies:

(a)

$$\nabla(au + bv) = a\nabla u + b\nabla v$$

(b)

$$\nabla(uv) = u\nabla v + v\nabla u$$

(c)

$$\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$$