MA134 Geometry and Motion

Examples Sheet 5

Answers to Part B questions must be handed in to your supervisor via the pigeon loft by 2pm Thursday, 14th February, 2019 (week 6).

Part A. Easier and background questions to be done first. Not to be handed in for marking.

1. Verify the following linear approximations at $(0,0)$

$$(a) \quad \frac{2x+3}{4y+1} \approx 3 + 2x - 12y \qquad\qquad (b) \quad \sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$$

2. Consider the function

$$f(x,y) = y^2 - x^2$$

Sketch contours of f in the region $0 \leq x \leq 2$, $0 \leq y \leq 2$. Show steepest descent vectors at the following points: $(1, 1/2)$ $(1,1)$ $(1/2, 1)$

3. Find the equation for the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

4. Consider the following nonlinear ordinary differential equations

$$\dot{x} = f(x,y) = \tan(x^2 + y) \qquad \dot{y} = g(x,y) = \tan(y^2 - x)$$

Approximate each of the right hand sides by a linear approximation about $(0,0)$ to obtain a system of linear ODEs. You should be able to solve these ODEs for initial conditions $x(0) = 0.1$, $y(0) = 0$.

5. Find and classify the critical points of

$$f(x,y) = 4 + x^3 + y^3 - 3xy$$

Suppose this f was used to generate a system of ODEs

$$\dot{x} = -\frac{\partial f}{\partial x}(x,y) \quad \dot{y} = -\frac{\partial f}{\partial y}(x,y)$$

Based on the above analysis of critical points, where would the initial condition $x = y = 1/2$ go as $t \to \infty$? (You don't need to calculate anything here.)

B. Questions for credit

6. Find all points at which the direction of steepest descent of the function

$$f(x,y) = x^2 + y^2 - 2x - 4y$$

is $(1/\sqrt{2}, 1/\sqrt{2})$. Identify the shape of contour lines and sketch at least three contours $\mathcal{L}_k$ of f with equal spacing between the contour levels k. Indicate the set of points you found on the sketch.

7. Find all points belonging to the ellipsoid

$$E = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$$

such that the normal to E at these points forms equal angles with the coordinate axes.
Hint. If $\mathbf{n}$ is a normal to E at such a point, then $\mathbf{n} \cdot \mathbf{i} = \mathbf{n} \cdot \mathbf{j} = \mathbf{n} \cdot \mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the direction of coordinate axes.

8. Two surfaces are called orthogonal at a point of intersection if their normals are perpendicular at that point. Show that the round cone $z^2 = x^2 + y^2$ and the unit sphere $x^2 + y^2 + z^2 = 1$ are orthogonal at every point of intersection. Sketch the surfaces and identify the set of intersection points.

9. Assume that the distribution of pressure in space is given by $p(x, y, z) = F(x^2 + y^2 + z^2)$, where F is a differentiable function of one variable. ("Spherically symmetric" distribution of pressure.) Show that a particle in space evolving according to the ordinary differential equation

$$\mathbf{r}'(t) = -\nabla p(\mathbf{r}(t))$$

either remains stationary or moves in the radial direction (to or away from the origin). Find the trajectory of the particle if $F(u) = u$ and $\mathbf{r}(0) = \mathbf{R} \in \mathbb{R}^3$.

10. A toy model for a drop of paint spreading on an inclined plane is given by level sets of the family of functions $u_t : \mathbb{R}^2 \to \mathbb{R}, t \geq 1$ such that

$$u_t(\mathbf{x}) = u_t(x, y) = 2A(t)\frac{y - x}{1 + x^2 + y^2},$$

where $A(t) = t^{1/2}$. The edge of the drop at each time $t \geq 1$ is the $k = 1$ level set $\mathcal{L}_1$ of u_t. Describe these time-dependent level sets. Sketch the drop at a few times starting from $t = 1$. Use your intuition to find the direction of the incline (the direction of the steepest descent along the plane).

C. More exercises.
(They should not be handed in for marking.)

12. Consider the following nonlinear ordinary differential equations

$$\ddot{x} = f(x, y) = -\frac{x}{4}e^{-\frac{x^2}{8}-\frac{y^2}{2}} \qquad \ddot{y} = g(x, y) = -ye^{-\frac{x^2}{8}-\frac{y^2}{2}}$$

These are difficult to solve. You should approximate each of the right hand sides by a linear approximation about $(0, 0)$. This will give you two linear ODEs. You should then give the general solution to these equations.

13. Find and classify all the critical points of the function

$$f(x, y) = y^2 - 2y\cos x, \quad \text{where} \quad 1 < x < 7$$

Then based on this analysis, sketch the contours of the function. Check the contours with Matlab or other software.

14. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f there, and if so how?

15. Show that the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at point (x_0, y_0, z_0) can be written

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$