MA134 Geometry and Motion

Examples Sheet 7

Answers to Part B questions must be handed in to your supervisor via the pigeon loft by 2pm Thursday, 28th February, 2019 (week 8).

Part A. Easier and background questions to be done first. Not to be handed in for marking.

1. Write the following equations in cylindrical coordinates

$$(a) \quad z = x^2 + y^2 \qquad (b) \quad x^2 + y^2 = 2x$$

2. Using integration in appropriate coordinates, derive the formula for the volume of a cylinder of radius R and height h, and derive the formula for the volume of a sphere of radius R.

3. Evaluate $\iiint_\Omega \sqrt{x^2+y^2}\, dV$ where Ω is the region inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

4. Integrals of the following form occur frequently in spherical coordinates $\int_c^\infty r^n e^{-r/a}\, dr$, where n is a positive integer, $c \in \mathbb{R}$ and a is a positive constant. Call this integral I_n and show that

$$I_n = ac^n e^{-c/a} + naI_{n-1}$$

and hence that

$$\int_c^\infty r^n e^{-r/a}\, dr = ae^{-c/a}\left(c^n + nac^{n-1} + n(n-1)a^2c^{n-2} + \cdots + n!a^n\right)$$

B. Questions for credit

5. Passing to polar coordinates, evaluate

$$\int\int_S \sqrt{a^2 - x^2 - y^2} dxdy,$$

where the domain S is bounded by the right loop of the lemniscate

$$(x^2+y^2)^2 = a^2(x^2-y^2),\ x \geq 0,$$

a is a positive constant.

6. Sketch regions whose areas are expressed by the integrals

$$(a) \quad \int_{\pi/4}^{\arctan 2} d\phi \int_0^{3\sec\phi} rdr; \qquad (b) \quad \int_{-\pi/2}^{\pi/2} d\phi \int_0^{a(1+cos\phi)} rdr.$$

Compute these areas. **Hint.** Examine the regions closely before deciding whether to evaluate the above integrals directly or to use different coordinates.

7. Evaluate

$$\int\int\int_D zdxdydz,$$

where D is bounded by the upper half of the cone $z^2 = \frac{h^2}{R^2}(x^2+y^2),\ z \geq 0$, and the plane $z = h$. Here h, R are positive constants. **Hint.** Use cylindrical coordinates.

8. Passing to spherical coordinates, evaluate the integral

$$\int\int\int_D \sqrt{x^2+y^2+z^2}dxdydz,$$

where D is the interior of the sphere $x^2+y^2+z^2=x$.

9. Compute the volume of a solid bounded by the sphere $x^2+y^2+z^2=a^2$ and the cone $z^2=x^2+y^2$ (external to the solid cone $z^2 \geq x^2+y^2$). Here a is a positive constant.

C. More exercises.
(They should not be handed in for marking.)

10. Sketch and find the volume of the solid bounded above by $z^2=x^2+y^2$, below by the xy-plane, and on the sides by $x^2+y^2=2ax$

11. Sketch and find the volume of the solid bounded above by $z=a-\sqrt{x^2+y^2}$, below by the xy-plane, and on the sides by $x^2+y^2=ax$

12. Consider a cylindrical antenna of radius R and length L. Assume the intensity of the radiation field inside the antenna is proportional to the square of distance from the cylindrical outer wall and proportional to $\sin^2(2\pi z/\lambda)$, where the z coordinate aligns with the cylinder axis and λ is the wavelength of the radiation ($\sim 12cm$ for Wifi). Find the total energy inside the antenna as the integral of the intensity.